DOCUMENT RESUME

ED 148 622 SE 023 749

AUTHOR	Dexter, John H.
TITLE	The Development of a Product for the Concrete Manipulation of Negative Numbers.
PUB DATE	75
NOTE	155p.; Ed.D. Dissertation, Teachers College, Columbia University; Contains occasional marginal legibility in Figures
EDRS PRICE	MF-$0.83 HC-$8.69 Plus Postage.
DESCRIPTORS	Curriculum; Doctoral Theses; *Educational Research; Elementary Education; *Elementary School Mathematics; Instruction; *Integers; *Manipulative Materials; *Mathematical Models; Mathematics Education
IDENTIFIERS	Cuisenaire Rods; *Negative Numbers

ABSTRACT

This study concerns concrete manipulatives for use in mathematics education, specifically in the realm of negative numbers and the operations on them. The author presents a review of relevant theory, practice, and research and develops guidelines regarding characteristics of manipulative materials. He concludes that present practice in teaching negative numbers does not use materials that meet the implications of theory. The study continues with an examination of the theory and research pertaining to the Cuisenaire rods which leads the author to accept them as a viable product capable of being extended into the set of negative integers. He then presents the design of a new product for negative numbers that is an extension of the rods and affirms the validity of the design by measuring it against the previously discussed related implications of theory. (MN)

****
* Documents acquired by ERIC include many informal unpublished *
* materials not available from other sources. ERIC makes every effort *
* to obtain the best copy available. Nevertheless, items of marginal *
* reproducibility are often encountered and this affects the quality *
* of the microfiche and hardcopy reproductions ERIC makes available *
* via the ERIC Document Reproduction Service (EDRS). EDRS is not *
* responsible for the quality of the original document. Reproductions *
* supplied by EDRS are the best that can be made from the original. *
****

ED148622

U.S. DEPARTMENT OF HEALTH,
EDUCATION & WELFARE
NATIONAL INSTITUTE OF
EDUCATION

THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL NATIONAL INSTITUTE OF EDUCATION POSITION OR POLICY

THE DEVELOPMENT OF A PRODUCT FOR THE CONCRETE MANIPULATION OF NEGATIVE NUMBERS

by

John Harry Dexter

"PERMISSION TO REPRODUCE THIS MATERIAL HAS BEEN GRANTED BY
John H. Dexter

TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC) AND USERS OF THE ERIC SYSTEM."

Dissertation Committee:

Professor Anne Marie Mueser, Sponsor
Professor Dwayne Huebner

Approved by the Committee on the Degree of Doctor of Education

Date ______________________

Submitted in partial fulfillment of the
requirements for the Degree of Doctor of Education in
Teachers College, Columbia University

1975

SE 023 749

© John Harry Dexter 1975

All Rights Reserved

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS iii

INTRODUCTION. 1

Chapter

I. THREE THEORETICAL PERSPECTIVES 4

Piaget -- The Developmental 4

Dienes -- The Pedagogical 9

Bruner -- The Curricular. 15

Summary 18

II. CURRENT PRACTICE IN THE TEACHING OF NEGATIVE NUMBERS. 25

Proofs. 27

Verbal Methods. 30

Summary. 34

Iconic Methods. 36

Summary. 50

Physical Methods. 51

Summary. 57

Summary 58

III. THE CUISENAIRE RODS: THEORY AND RESEARCH 65

Introduction. 65

Theory 67

Summary. 74

Research. 75

Introduction 75

Survey. 79

Summary 97

IV. A CONCRETE MANIPULATIVE FOR TEACHING NEGATIVE NUMBERS 98
  Introduction 98
  Operations 101
  Relationship to Theory 112
V. GUIDELINES FOR TEACHERS 118
  Introduction 118
  Free-play. 122
  Games. 123
  Dialogue 124
  Session 1. 124
  Session 2. 125
  Session 3. 127
  Session 4. 128
  Session 5. 130
  Symbols. 132
  Session 6. 132
  Session 7. 134
  Session 8. 135
  Session 9. 137
  Recommendations. 138
  Conclusion 139
BIBLIOGRAPHY. 141

LIST OF ILLUSTRATIONS

Figure

1. Graph, $^{-}3 \times {}^{-}4 = {}^{+}12$ 42

2. $^{+}3$ Multiplier Line 44

3. $^{-}4$ Multiplier Line 45

4. Graph, $^{-}4 \times {}^{-}2 = {}^{+}8$ 47

5. Negative Rods and Cuisenaire Rods. 100

6. The Addition of Two Negative Quantities. 103

7. The Addition of a Positive and a Negative Quantity. 104

8. Subtraction from Zero. 106

9. Subtraction. 107

10. Subtraction. 108

INTRODUCTION

This study will present a new design for a concrete manipulative for use in mathematics education, specifically in the realm of negative numbers and the operations on them. A review of the literature on the theory of manipulative materials strongly suggests that mathematics education has been and can continue to be greatly enhanced through the employment of concrete manipulatives in the classroom. Inasmuch as the teaching of negative numbers forms a part of mathematics education it seems that that educational practice could likewise benefit from the introduction of an appropriate concrete manipulative to represent negative numbers.

At the present time the formal introduction of negative numbers, especially in the operations of multiplication and division takes place in a pre-algebra class at about the sixth or seventh grade. Regardless of the grade level, the introductions to negative numbers use various techniques to try to make available the universe of negative numbers and operations on them. Various representations are presented in formats that include the use of stories, pictures, or physical objects to attempt to connect the operations on negative numbers to life-like operations in the real world. Other kinds of introductions rely on mathematical proofs which are themselves valid and clear, but are not always accepted by students as being reasonable. The product of $(^-1) \cdot (^-1)$ is often accepted without reason by students due to their inability to imagine that the product of two negative factors should be positive. In the absence of experience, many students either blindly accept the product, or begin to disregard mathematics as a logical system. Clearly, an appropriate concrete manipulative could provide the necessary experience which will obviate this problem.

In order to aid in the development of some guidelines regarding the desired characteristics of a concrete manipulative this study will include a presentation and review of certain theoretical propositions. These ideas will then be compared with current practice in the teaching of negative numbers to determine to what extent the materials and methods currently in use reflect the implications of theory. Are the materials and methods currently in use appropriate, not only for representing negative numbers, but as concrete materials themselves? Does their design aid in the teaching of negative numbers?

The design of the new product for negative numbers will not only reflect the implications of the theories but will be so structured that it becomes an extension of a concrete manipulative already in use, the Cuisenaire rods. The rods have been the topic of certain theoretical discussions and their application in the classroom has been reported on and researched. Consequently, and in order to justify the rods as being worthy of extension, this study will present summaries of the theory and research regarding the rods. The implications of the available information are sufficiently positive to permit the extension of the rods into the realm of negative numbers. Further, the theory and research to be surveyed will serve to familiarize the reader with the uses of the rods in the four fundamental operations on positive numbers (addition, subtraction, multiplication, and division).

With the establishment of the viability of Cuisenaire rods for positive numbers, the design for a concrete manipulative for negative numbers which extends the rods can be seen as a valid one. In order to further affirm this validity, the design will be measured against the implications of the theory relating to it. The presentation of that theory forms the content of chapter one.

Chapter I

THREE THEORETICAL PERSPECTIVES

Piaget -- The Developmental

Jean Piaget's theory of stage development is well known to early childhood education specialists, learning theorists, and mathematics educators. He postulates four stages of development each of which is entered and eventually exited due to four factors in the child's life. The four stages and their approximate age ranges are:

1) sensori-motor; birth to two years
2) pre-operational; 2 years to seven years
3) concrete operational; 7 years to eleven years
4) formal operational; eleven years ---[1]

The four factors which bring about transition from stage to stage are:

> the innateness of the structures in the nervous system, physical experience, social transmission, and the probabilistic laws of equilibrium.[2]

These four factors provide an explanation of the transition from an "... insensibility to the most simple deductions to the specific state of awareness which characterizes logical necessity..."[3] Piaget continues with some considerations of how the four factors interact. Regarding the innate structures he states:

---

[1] Jean Piaget, Six Psychological Studies, trans. Anita Tenzer, ed. David Elkind (New York: Vintage Books, 1967), pp. 122-124.

[2] Ibid., p. 127.

[3] Ibid.

> ... while the coordinations of the nervous system determine the framework of possibilities and impossibilities within which the logical structures will be constructed, these logical structures do not exist within the nervous system as embryonic instruments of thought.[1]

In order to explain the eventual appearance of logical structures Piaget talks about actions as the basis of logical structures and the conforming of physical structures to the logical structures.[2] The source of logical operations then is action on physical objects, the laws of which are in line with logical structures. The social factor relates to the interchange which takes place among persons and the coordination of their actions at the interpersonal level. This takes place, according to Piaget, "... through common work and verbal exchange."[3] The fourth factor, equilibrium, "... can be conceived as a compensation for external intrusion by means of the activities of the subject, which are responses to these intrusions."[4] It can be seen that this factor is also operative within the first three factors identified. Piaget states:

> ... the social coordination of actions involves both disequilibria and forms of equilibrium and ... the interactions among the personal (neurological, etc.) factors and the social factors of action stem from a continual equilibration.[5]

---

[1] Ibid.

[2] Ibid., pp. 127-129.

[3] Ibid., p. 129.

[4] Ibid., p. 113.

[5] Ibid., p. 130.

In the area of physical experience, reversibility is closely related to the notion of equilibrium. For Piaget, "Whether we say that the operations are organized into reversible structures or that they tend toward certain forms of equilibrium ... amounts to the same thing."[1] Reversibility develops gradually during the mental growth of the child. At the sensorimotor level there is only a slight awareness of "... a practical reversibility in proximal space."[2] The preoperational subject has only a "semi-reversibility" related to compensations of gross errors. At the operational level the subject is gaining control of two forms of reversibility; in the operations of classes, the subject has inversion or negation; in the operations of relations, he is gaining control of reciprocity. The final stage, formal operational, is characterized by a single system of reversibility through the establishment of inversions and reciprocities in all operations and structures.[3]

With the above theory in mind, we can proceed to a more detailed examination of how Piaget views the growth of the child's awareness of mathematical structures and operations. Using the example of transitivity (e.g. A=C if A=B and B=C), Piaget notes that there is a level at which the child will not accept this relation without seeing it in concrete form. It is not until the age of seven or eight that the child will use deduction to accept either transitivity or commutativity (e.g. A+B=B+A) and the knowing of the child which allows acceptance of the relations began through experience.[1]

---

[1] Ibid., p. 131.

[2] Ibid., p. 130.

[3] Ibid.

The experience which the child has had may have been at one of two levels, the physical, or the logico-mathematical. Both of these experiences are obtained through work on physical objects but the subject attends to different aspects of the experiences such that one kind of attending produces a physical experience while another produces a logico-mathematical one. The physical experience is one in which the subject is attending to the objects at hand. "It consists in acting on objects to extract knowledge by abstraction based on these objects themselves."[2] This kind of experience is limited, to some extent, by the objects themselves. By attending only to the objects, the subject has diminished the possibility of transferring the extracted knowledge to other objects.

> The logico-mathematical experience ... consists in acting on objects, but with abstraction of knowledge based on action and no longer on objects themselves.[3]

By abstracting knowledge from action, the subject gains a more generally applicable experience such that he can be attending to new characteristics which may or may not be present in the objects themselves. That is, inasmuch as he is abstracting from the objects to the operations, he is no longer bound by the physical characteristics of the objects. Instead, he is introducing new characteristics through his abstracting and the experience is focussed on the interconnections among the characteristics.

---

[1] Jean Piaget, Psychology and Epistemology, trans. Arnold Rosin (New York: Viking Press, 1971), p. 70.

[2] Ibid.

[3] Ibid., p. 71.

> This is why the logico-mathematical actions of the subject can, at a specific moment, avoid being applied to physical objects and can be interiorized in symbolically manipulated operations. In other words, that is why, beginning at a certain level, pure logic and pure mathematics exist, for which experience becomes useless. That is why, moreover, this pure logic and pure mathematics are forever capable of transcending physical experience, since they are not limited by the physical characteristics of the object.[1]

These ideas suggest the need to highlight three points relevant to mathematics education. 1) Abstraction is available to a subject at the level of physical experience. In contrast to much opinion, Piaget does not say abstraction is not available to the young child. On the contrary, the child abstracts while manipulating physical objects. It is, however, limited abstraction since it is concerned only with the operations available with the physical objects and not the relations among the operations. 2) The second point, then, is that in order to reduce the limits of this attention to the physical, it becomes helpful for a teacher to direct the attention of the child to his actions and to provide a variety of physical objects so that the results of his actions can be seen to be consistent across different objects. 3) The problem of reliance upon the physical objects will not arise if the attention of the subject is focussed upon action. If the four factors which Piaget identifies are operating then the moment of transcendence from the physical will come about.

---

[1] Ibid., pp. 71-72. See also, Jean Piaget, Genetic Epistemology, trans. Eleanor Duckworth (New York: Columbia University Press, 1970), pp. 15-19.

Dienes -- The Pedagogical

Another theorist who has worked on mathematics learning is Z. P. Dienes. In addition to his work on learning, Dienes has written about teaching and has developed concrete manipulatives for use in the mathematics classroom. For Dienes,

> ... mathematics is based on experience; it is the crystallization of relationships into a beautifully regular structure, distilled from our actual contacts with the real world. Logic consists in reflecting on how this and other structures function, and I find it difficult to see how it is possible to reflect on something that is not yet there.[1]

What should be there is a concrete manipulative with which to experience the attributes of structures. The manipulative should be so structured that the experiences will be specifically related to the structures of mathematics and logic.[2] When such concrete materials are available, Dienes sees four principles which must guide the provision for experiences.

---

[1] Z. P. Dienes, Building Up Mathematics (London: Hutchinson Educational, 1960), p. 11.

[2] Ibid., p. 27.

1. Dynamic Principle. Preliminary structured and practice games must be provided as necessary learning experiences from which mathematical concepts can eventually be built, so long as each type of game is introduced at the appropriate time. Although while children are young these games must perforce be played with concrete material, mental games can gradually be introduced to give a taste of that most fascinating of all games, mathematical research.

2. Constructivity Principle. In the structuring of the games, construction should always precede analysis, which is almost altogether absent from children's thinking until the age of 12.

3. Mathematical Variability Principle. Concepts involving variables should be learnt by involving the largest possible number of variables.

4. Perceptual Variability Principle. To allow as much scope as possible for individual variations in concept-formation, as well as to induce children to gather the mathematical essence of an abstraction, the same conceptual structure should be presented in as many perceptual equivalents as possible.[1]

Principles 1 and 2 refer to the need for experience. This need for experience is expressed in two principles in order to differentiate between two initiators of such experiences. Principle one refers to those experiences to be directed by the child, (play). The second principle refers to those experiences that are more planned, that is, directed by the teacher. The third principle refers to the need for a large number of experiences relating to a specific concept. For example, if we are concerned with the transitive relation, we should design a large number of experiences which will allow the child to see that when A=B and B=C, then, A=C. The fourth principle calls for a variety of physical objects for two reasons: 1) the child will have the opportunity to note the consistency of the mathematics involved in similar manipulations of differing materials; and, 2) the variety of materials will permit a child to work with those to which he responds most easily.

---

[1] Ibid., p. 44.

The four principles are meant to guide a series of experiences which will cause the child to develop and employ some of the basic processes of mathematics learning. These processes have been identified by Dienes as: Abstraction, Higher Level Abstractions, Generalization, Particularization, Symbolism, and Interpretation.[1] The first four processes are concerned with structures, the last two with language. The processes that have to do with the structural aspects of mathematics learning are all preceded by play. Play is a "... tentative exploring activity..."[2] whose purpose seems to be to find the limits of the environment, in the case of mathematics learning, the limits of the uses for the physical material. Dienes' definition of abstraction is not identical to Piaget's. For Dienes, abstraction is a combinatorial act of drawing out from a situation that which is common to other situations from which we could also draw out.[1] Abstraction in this definition, is the act of drawing out with consciously applied criteria. Piaget does not include the use of criteria in his definition of abstraction. Higher-level abstractions are possibilities that arise when the "... abstractions resulting from one cycle ... provide a basis for the next, higher order, cycles."[2] Abstracting, for Dienes, results in the creation of a class from which we can abstract further (Higher-Level Abstractions). Using the defined class created by abstracting we can extend it to a wider class which has the same or similar properties. This is generalization. The difference between abstraction and generalization is that abstraction is not reversible while generalization is reversible. In Dienes' words:

---

[1] Z. P. Dienes, "Some Basic Processes Involved in Mathematics Learning," in Research in Mathematics Education, ed. National Council of Teachers of Mathematics (Washington, D.C.: National Council of Teachers of Mathematics, 1967), pp. 21-34.

[2] Ibid., p. 21.

> Once a class has been created, it is inconceivable that it can be uncreated. The generalization process can be reversed. It is equally possible to pass from a more to a less extensive class and to pass from a less to a more extensive class.[3]

The act of going from a more extensive class to a less extensive class is called, by Dienes, particularization.[4] Particularization refers to restrictions which may be applied to operations within a more extensive class. These restrictions serve to create a less extensive class wholly contained by the more extensive one. For example, a two dimensional graph will permit the plotting of certain ordered pairs of numbers (depending upon the size of the graph). If we choose to, we can assign a rule to regulate the ordered pairs such as; the sum of the pairs must be zero. This rule particularizes the class of ordered pairs such that the first number of the pair must be the opposite of the second number.

---

[1] Ibid., p. 22.

[2] Ibid., p. 24.

[3] Ibid., p. 26.

[4] Ibid.

The relationships among the four structural processes and the four principles might be established by viewing the four structural processes as having two classes, elemental and secondary. The elemental processes are Abstraction and Generalization. The secondary processes, which grow out of the elemental ones, are Higher-Level Abstractions and Particularization. The elemental processes determine the kinds of experiences that should be provided for the mathematics student. After the initial play period, it will be necessary to provide a large number of different situations using different materials in order to encourage the learner to abstract in the way abstraction has been previously defined. These experiences shall be guided by the first two of Dienes' principles, Dynamic and Constructivity. Also, the fourth principle, Perceptual Variability can be seen as relevant for it takes into account varieties of learning styles as well as multiple experiences. For Generalization, the relevant principle is Mathematical Variability which will direct the provisions for numerous experiences related to a specific, generalizable, concept.

At some point within the experiences - cum - processes the need for language will arise. Symbolism is the form by which mathematical situations are represented and therefore symbols taken together form the code of mathematical language.[1] The problem of mathematical language then is equivalent to the problem of encoding and decoding. For Dienes, encoding is Symbolism, decoding is Interpretation.[2] Dienes discusses the problems of symbolism and notes that there has been little research into the problem of what meaning a symbol has to a child, when to introduce the symbol, and to what extent the ability to manipulate symbols is reflexive to mathematical understanding. He concludes, tentatively, that it is probably best to introduce symbols after a variety of concrete experiences.[3] Interpretation, the problem of decoding, is the reverse of symbolization in that the learner is now concerned with knowing a situation that has been represented with a symbol. Dienes states: "... symbolization and its interpretation should, perhaps, be emphasized in schools and experimented with in psychological laboratories."[4]

---

[1] Ibid., pp. 26-28.

[2] Ibid., pp. 28-29.

[3] Ibid., pp. 27-28.

[4] Ibid., p. 28.

In retrospect, we can see the interrelation of Dienes' principles with the processes he has identified. He does not present a developmental sequence. Instead, his concern is with the organization of mathematical experiences, the sequence and scope, in other words, the learning cycles which should take place in order for mastery of each mathematical idea.

Bruner -- The Curricular

Jerome Bruner, a psychologist who has suggested a theory of instruction, has identified four major features to be kept in mind while designing learning environments and experiences. According to Bruner a designer should, 1) specify experiences; 2) find ways of structuring bodies of knowledge; 3) establish effective sequences, and; 4) determine the nature of pacing rewards and punishments.[1] The second of these features, structuring bodies of knowledge, is important for this study since it speaks to

> ... the ways in which a body of knowledge should be structured so that it can be most readily grasped by the learner.[2]

Bruner describes the attributes of useful structures:

> ... the merit of a structure depends upon its power for simplifying information, for generating new propositions, and for increasing the manipulability of a body of knowledge.[3]

---

[1] Jerome S. Bruner, *Toward a Theory of Instruction* (New York: W. W. Norton, 1968), p. 41.

[2] Ibid.

[3] Ibid.

These three attributes may be summarized as mode, power, and economy. Economy refers to how much information must be kept in mind by a learner in order to arrive at a conclusion regarding the data. The more information necessary to keep in mind the lower the economy and therefore the manipulability of the body of knowledge.[1] Power refers to the generative value of a set of learned propositions. When a structure has power, one can expect a learner to make connections between ideas and propositions which have not been specifically presented to him.[2] Mode refers to the three ways in which a body of knowledge may be represented. Bruner states:

> Since most subjects can be translated into forms that place emphasis upon doing, or upon the development of appropriate imagery, or upon symbolic-verbal encoding, it is often possible to render the end result to be achieved in a simpler, more manageable form so that the child can move more easily and deeply to full mastery.[3]

These three forms of representation are referred to as the enactive, iconic and symbolic modes. An example of an application of these modes to curriculum may be found in Man: A Course of Study.[4] In one section of Man: A Course of Study the children are asked to construct a model of an African savannah, they then view movies of a baboon troop on the savannah, and finally they read from an anthropologist's field notes on his observations of life on the savannah. This sequence, when applied to mathematics, calls for an enactive stage when the learner is building with concrete materials, an iconic stage when the learner is representing his building with images, and a symbolic stage when the learner, in dialogue with the teacher, establishes a set of symbols which will stand for the materials and their assembly. For example, a young child might build a wall using colored inch cubes. He might then be asked to draw a picture of his wall. Depending upon his age and experience, he might be required to count precisely the number of cubes at each level of his wall and represent them appropriately, or, he might be asked to match the colors in his drawing to the colors in the wall. When his drawing is complete, some symbolic representation will be found, possibly using the first letters of the color names, to allow easier recording of his wall. Bruner's ideas for the curriculum, generally, are applicable to mathematics. He describes mathematics learning as a threefold process, progressing through each of the three modes of representation. The initial activity, an instrumental one, accomplishes definitions of things by doing things. The instrumental operations are then summarized and represented in the form of images. This process allows the learner to understand the formal abstract properties of the mathematical ideas he is working on. The learner's understanding is aided by the formal notation which is constant over varying transformations of imagery. For Bruner, this abstraction is a freeing for the learner from the physical limitations of the things he has worked with. At the same time, though, he has built a repertoire of images with which he can clarify new situations.[1] Other writers have expressed similar ideas regarding the building of images.

---

[1] Ibid., p. 45.

[2] Ibid., pp. 47-48.

[3] Ibid., p. 30.

[4] Education Development Center, Man: A Course of Study, vol. 5, Baboons (Washington, D. C.: Curriculum Development Associates, 1970).

W. Servais states:

> ... every perception or action derived from the concrete duplicates itself in mental imagery; this becomes structured and can then be recalled in its own right.[2]

For Servais, these perceptions and ideas evolve out of a dialogue with the concrete materials; and the interrelations between perceptions and actions which exist at the level of images, are the first objects for mathematical study. He states:

> There is no real action which does not duplicate itself in a mental representation which becomes more and more conscious and increasingly distinct from that action. This representation in the mind is already a spontaneous abstraction since it contains only a simplified and rapidly stereotyped view of the concrete procedure. ... The mental representation of a succession of virtual actions, the realisation of how they arrange themselves in a certain order and how they allow of being inverted, is the approach to order structures and algebraic structure.[3]

Summary

Servais' ideas can be read as a summary of what we have seen from our survey of the theoretical ideas regarding manipulatives; that concrete manipulation acts as preparation for the mental manipulations of symbols in mathematics education. For Piaget, the child abstracts from his action on the physical material to develop the ability to transfer his knowledge (structures) to other situations. Dienes, not in opposition to Piaget, says that the initial play experiences with the physical material provides the basis from which the basic processes grow. His theory of abstraction sees that act as more complex than does Piaget. Bruner, likewise, sees abstraction as somewhat more complex than does Piaget. Abstraction for him follows early symbolizing and does not seem to flow directly from physical activity. Servais, like Bruner, sees the power of physical manipulation as preparation for abstract thinking, but abstraction for Servais takes place spontaneously in the virtual act of thinking, in stereotypical images, through manipulations.

---

[1] Bruner, Toward a Theory of Instruction, p. 68.

[2] W. Servais, "The Significance of Concrete Materials in the Teaching of Mathematics," in Mathematical Reflections, ed. Association of Teachers of Mathematics (Cambridge, England: Cambridge University Press, 1970), p. 207.

[3] Ibid., p. 208.

While there are some differences in the above views regarding what abstraction is, and when it is available to the learner, all of the above thinkers are in agreement regarding the importance of the concrete experience.

Further evidence of the importance of the concrete experience can be seen when we compare some of the points each writer makes regarding the factors obtaining in mathematical learning. The following chart will serve to clarify the comparisons.

	Piaget	Dienes	Bruner
1.	Physical experience	Dynamic principle	Definition by doing
2.	Mental structures	Constructivity principle	Imaging
3.	Social transmission	Variability principles	Notation

The factors in row one refer to a student's earliest experiences which relate to mathematics. For Piaget, these experiences should be in the form of action on physical objects; for Dienes they are games and practice with concrete materials to provide learning experiences which relate to mathematical concepts. Bruner talks about the action of the child as bringing about definitions. It can be seen that each theorist is concerned about play with concrete materials that provides experiences relevant to mathematics and logic.

These play experiences affect the factors in the second row: mental structures, the constructivity principle, and imaging. Each of these phrases refers to the mental set of the learner; the way in which he relates to his experiences. For Piaget, we have seen that the physical laws relevant to the manipulation of objects are similar to the laws of logic and that these laws are learned through action. Dienes' second principle makes reference to the need for the building situations long before the analysis of them. Bruner's imaging is in reference to the stock of images which the child is creating as he works with the material.

The experiences which the learner is having are eventually guided by the factors in row three. In the items referred to, each theorist is calling upon the teacher, or the one who is designing the environment, to begin to transmit, vary the experiences with, or represent symbolically the traditions of the discipline. While this set of factors is not explicitly related to manipulative materials, taken in the context of the prior two, it can be seen to be related to them. Each of the above theorists, from a different stance, (the psychological, the pedagogical, and the curricular) has established, theoretically, the need for physical manipulatives in mathematics teaching and learning.

The attributes which a concrete manipulative should have have also been defined by these theorists, although not completely. In the realm of positive numbers, more specifically the counting set, almost any countable physical object will serve as a manipulative inasmuch as its countability will ensure its relationship to the mathematics available. As Piaget has pointed out, the source of the logical structures is the action on the physical; the attention of the actor, whether toward the material or the action itself, defines the level of experience obtaining, physical or logico-mathematical.[1] When physical manipulatives are available, they should be simple enough in structure and meaning so that the student's attention may be focussed on the action, rather than be caught up in the problems of definitions regarding the materials.

---

[1] Piaget, Psychology and Epistemology, trans. Arnold Rosin, p. 15. See chap. I, p. 7, above.

Dienes' design for mathematics education takes for granted the presence of concrete manipulative materials. He has stated:

> Structured materials ... are to be thought of as filling the gap in the children's concept learning where actual experiences in life cannot be provided.[1]

He has established four principles and identified some of the basic processes which, when taken together, serve to guide the experiences of the mathematics student.[2] From his principles we know that the materials and situations used in mathematics teaching should be varied. From the processes he identified we know that play is an important precursor to abstraction and all that follows it. It is not necessary for a single manipulative to fulfill the principle relating to variation. It is, however, necessary for that manipulative or method to be useable in a variety of situations with differing content. In regard to play, it is clear that, like Piaget, Dienes expects a concrete manipulative to carry with it the properties also found in mathematics and logic. Where this is not true, play will not be possible, or will not prepare the player for the logic to be explored.

---

[1] Z. P. Dienes, "Multi-base Arithmetic," in Problems in the Teaching of Elementary School Mathematics, ed. Klaas Kramer (Boston: Allyn and Bacon, 1970), p. 308.

[2] Dienes, Building Up Mathematics, p. 44; Dienes, "Some Basic Processes Involved in Mathematics Learning," pp. 21-34. See chap. I, pp. 11-15, above.

Bruner's outline for mathematics education also includes concrete manipulatives especially within the enactive mode. He has also identified three factors which should be taken into account when structuring subject matter: mode, power, and economy.[1] Further, he notes that knowledge should be structured in a way which makes it easy for the learner to grasp.[2] This is an important consideration when a methodology is being designed for use with students.

The structure of the knowledge generated by negative numbers and the operations on them has taken a variety of forms because the set of negative numbers is difficult to represent. Some mathematics educators have chosen to introduce the topic by way of formal proofs. This may be satisfactory for certain twelve year old students who have begun to have the ability to comprehend and use formal operations. Piaget states:

> These are the operations of "logical propositions," which from this time on, can deal with simple verbal statements (propositions), i.e., with simple hypotheses rather than exclusively with objects.[3]

Nevertheless, the final phrase of the above statement is an indication that concrete materials can continue to help the student as he begins to enter the stage of formal operations. Therefore a concrete manipulative would not be redundant for these students.

---

[1] Bruner, Toward a Theory of Instruction, p. 45. See chap. I, pp. 15-18, above.

[2] Ibid., p. 41. See chap. I, p. 16, above.

[3] Piaget, Six Psychological Studies, trans. Anita Tenzer, ed. David Elkind, p. 125.

Along with the possibility for a concrete manipulative for teaching negative numbers there exists the possibility for other formats. Mathematics educators have designed a variety of methodologies for helping those students who have not achieved a level of operations which permits them to deal with hypotheses. A survey of some of these attempts forms the content of chapter two.

Chapter II

CURRENT PRACTICE IN THE TEACHING OF NEGATIVE NUMBERS

In chapter one some of the theories which speak to the use of concrete manipulatives and how that use relates to mathematics learning were discussed. In this chapter one topic within mathematics education, negative numbers, will be examined. Current practice in teaching negative numbers will be described and the nature of that practice will be compared with the theory presented in chapter one to determine whether it has been informed by that theory.

According to Dienes, structured materials are objects which will fill gaps where real-life situations cannot be provided. Similarly, Elizabeth Fennema has said:

> Manipulative materials can help to provide the real situations that enable a child to understand and effectively use symbols.[1]

The study of negative numbers is a topic in mathematics education where real situations are difficult to provide. Helene Sherman, in her survey of new mathematics curricula, quotes and comments upon Florian Cajori's historical note regarding negative numbers.

> Cajori discusses the question of why it was so difficult to include the idea of the negative in the concept of number. Reasoning that until mathematicians "hit upon a visual or graphical representation of them," negative numbers seemed "absurd" or "fictitious," he comments on

---

[1] Elizabeth Fennema, "Manipulatives in the Classroom," Arithmetic Teacher 20 (May 1973): 350.

> the importance of such representations in the teaching of algebra. "Omit all illustrations by lines or by the thermometer," he warns, "and negative numbers will be as absurd to modern students as they were to the early algebraists."[1]

There have been a number of attempts to fulfill the need for visual or graphical representations of negative numbers and this chapter shall survey some of them.

Ashlock and West have identified two types of methods used to illustrate the properties of negative numbers:

> ... those based on measurement and those based on counting. Measurement ... refers to the use of units along a continuum, whereas counting refers to the use of discrete elements of sets.[2]

Along with this two-fold delineation of current practice we can identify the mode in which these types are represented. That is, by adapting Bruner's three modes of enactive, iconic, and symbolic representation[3] to modes called physical, iconic, and verbal representation, we will have created a classification which will aid the organization of our survey. Further, such a classification can help to differentiate among methods which are actually quite similar but, because of their content, appear to be different. This will serve to point out important differences among the methods and it might also help those interested in applying the methods in the classroom. That is, teachers will be able to avoid repeating a method-type which has already failed to enlighten a student.

---

[1] Helene Sherman, Common Elements in New Mathematics Programs (New York: Teachers College Press, Columbia University, 1972), pp. 89-90.

[2] Robert B. Ashlock and Tommie A. West, "Physical Representations for Signed-Number Operations," Arithmetic Teacher 14 (November 1967): 549.

[3] Bruner, Toward a Theory of Instruction, pp. 10-11.

A verbal model is one which uses language to tell a story or establish a situation which serves to cause the student to reflect such that he begins to deal with the operations of negative numbers. Generally it would be difficult to create a picture of the situation being described with verbal models because of directional difficulties. For example, drawing a picture of the act of giving a gift does not always make clear who is giving and who is receiving the object.

An iconic model for representing negative numbers uses some form of a picture or graph. It does not rely on the verbal alone, nor is it so structured that it is manipulable.

A physical model is one which may be manipulated physically by the learner in order to experience the operations of negative numbers.

There are two types of illustrations, counting and measuring, which are represented in three modes, verbal, iconic, and physical. This produces six different classifications for practice. That is, in each of the verbal, iconic or physical modes there may be a requirement to count or measure.

Proofs

The only forms of representation which are not included in the above classifications are the purely symbolic ones that are based on definitions or proofs. These presentations use characteristics of numbers already known to students to establish the operations on negative numbers. One example of this is that used by Wheeler[1] among others which includes the characteristics of zero in multiplication as well as the distributive law.

$$
\begin{aligned}
0 \cdot {}^{-}6 &= 0 \\
({}^{-}3+{}^{+}3) \cdot {}^{-}6 &= 0 \\
({}^{-}3 \cdot {}^{-}6)+({}^{+}3 \cdot {}^{-}6) &= 0 \\
({}^{-}3 \cdot {}^{-}6) + ({}^{-}18) &= 0 \\
({}^{+}18) + ({}^{-}18) &= 0
\end{aligned}
$$

Another popular method for defining the multiplication of negative numbers is cited by Peterson. His example is:

> The integer $^{+}2$ is defined as equivalent to the set of ordered pairs (a,b) such that $a-b={}^{+}2$ where a and b are whole numbers. So, $^{+}2 \equiv (2,0) = (3,1) = (4,2) = \ldots$ This definition represents $^{-}5$ by ordered pairs such as $(0,5)$, $(1,6)$, $(2,7)$, and so on.
>
> Multiplication is defined by $(a,b) \times (c,d) = (ac+bd,\ ad+bc)$. Thus $^{+}3 \times {}^{+}2$ becomes $(4,1) \times (5,3) = (4\times5 + 1\times3,\ 4\times3 + 1\times5) = (20+3,\ 12+5) = (23,17) \equiv {}^{+}6$; and $^{-}3 \times {}^{-}2 = (1,4) \times (3,5) = (1\times3 + 4\times5,\ 1\times5 + 4\times3) = (3+20,\ 5+12) = (23,17) \equiv {}^{+}6$.[2]

These two methods of defining the product of two negatives as a positive are clear definitions but they probably do not serve to establish "... the inner reasonableness of $({}^{-}1) \times ({}^{-}1) = {}^{+}1$."[3] This is where the major problem of understanding negative numbers lies, in the multiplication of them. No topic at the Cambridge Conference generated more discussion and a variety of approaches to the problem showed that the problem was a teaching one rather than a mathematical one.[1] Inasmuch as the teaching of multiplication of negative numbers is where most of the difficulties lie, this survey of practice will be concentrated on those attempts at establishing the reasonableness of this operation.

---

[1] Ruric E. Wheeler, Modern Mathematics: An Elementary Approach (Belmont, California: Brooks/Cole Publishing, 1970), pp. 151-152.

[2] John C. Peterson, "Fourteen Different Strategies for Multiplication of Integers or Why $({}^{-}1)({}^{-}1)={}^{+}1$," Arithmetic Teacher 19 (May 1972): 402-403.

[3] Cambridge Conference on School Mathematics, "Curriculum for the Elementary School (K-6)," in Problems in the Teaching of Elementary School Mathematics, ed. Klaas Kramer (Boston: Allyn and Bacon, 1970), p. 15.

Page[2] and later, Hall[3] have each written guidelines for using language in connection with the various possibilities for reading the (-) sign. Each writer recommends three words whose use is made clear from context. The words, "minus," "negative," and "opposite" are recommended for use in the following context: "minus" for subtraction, ex. 6-2; "negative" for a quantity less than zero, ex. ¯9; "opposite" for a variable, ex. -a; and when this writing is found -(¯20), read, "the opposite of negative twenty."

These uses are intended to clarify the sign to be found in a simplified expression. For example, the expression -(¯8-5) is made clearer if we decode the symbols as: "the opposite of negative eight minus five." We can simplify the parentheses and get -(¯13) and read again: "the opposite of negative thirteen." If we are aware of the meaning of opposite, it is clear that the value of the expression is $^{+}13$. Likewise with a variable, "a". When a variable is found in an expression it may be either above, equal to, or below zero and therefore be a positive, zero, or negative number. By reading the (-) sign found in front of the variable as: "the opposite of" we are maintaining in our minds the possibility of the variable's value as either positive or negative. When $a={}^{-}2$ then the opposite of a, written: ^{-}a, has the value positive two.

---

[1] Ibid.

[2] David A. Page, *Number Lines, Functions, and Fundamental Topics* (New York: Macmillan, 1964), pp. 136-138.

[3] Wayne H. Hall, "These Three: *Minus*, *Negative*, *Opposite*," *Arithmetic Teacher* 21 (December 1974): 712-713.

Verbal Methods

These attempts at building a way of reading for clarity of meaning are useful within the realm of symbolic proofs and definitions. The vocabulary is already available to most students, only the recognition of the context needs to be attended to. Other mathematics educators have seen the value of various ways to read signs for aiding understanding, and they have taken further steps by way of assigning more vocabulary to the expressions found when using negative numbers. These methods are examples of verbal representation of negative numbers. That is, they are models which do not lend themselves easily to pictures or diagrams. In most cases, an attempt at depicting the situation being evolved verbally would not be able to make clear the direction of the situation. This difficulty will be made clear as the various verbal representations are surveyed.

Skemp[1] begins his method for negative number operations by redesigning the signs of multiplication $\otimes$ and addition $\oplus$ reasoning that while the operation referred to by the sign is the same, the numbers in use are different. He also attaches words to each of the (+) and (-) signs so that they are read: (+), "do," and; (-), "undo," when found at the beginning of a multiplication expression. The second factor in a multiplication expression also has vocabulary for the signs found there: (+), "go up", and (-), "go down". The model to be kept in mind is the act of going up or down a set of steps and measuring the distance and direction travelled. It is therefore a model that is of the measuring type. The first factor informs us of how many times we should do or undo what the second factor says. For example: $^{+}3 \otimes {}^{-}2$, is read: "do three times, go down two steps." Thus, the multiplication of two negative integers might be: $^{-}1 \otimes {}^{-}5$, and would read, "undo once, go down five steps." Skemp's explanation of this situation is that "to undo an act, we have to do an equal and opposite act."[2] Therefore the above expression means the same as, "go up five steps." In order to understand negative number multiplication with Skemp's method it is necessary to understand the nature of the act of undoing on the set of steps. For many students it seems likely that undoing something that has been done might mean returning to the starting place. This would not be a useful understanding for Skemp's method. The student, using this method, must understand that the going down of five steps has been accomplished and that the expression tells him to perform an act which will undo a pre-existing situation. He is then expected to encode the kind of act necessary for the undoing.

---

[1] Richard R. Skemp, The Psychology of Learning Mathematics (Middlesex, England: Penguin Books, 1971), pp. 201-207.

[2] Ibid., p. 203.

Another mental, or purely verbal way of dealing with the dynamics of negative numbers uses stories about a postman who both delivers and takes back mail. The model was developed by Cohen[1] and later incorporated into a curriculum by Davis.[2] Again we have language applied to the (+) and (-) signs depending upon the context. Also, the factors and signs represent different entities. That is, the first sign refers to deliveries or take backs and the second sign refers to the content of the delivery, whether it had checks or bills. The first factor refers to the number of things being delivered or taken away. In other words, it answers the question, how many? The second factor refers to the amount of the checks or bills. This then is a method which uses counting to evaluate the situation.

For the first factor (+) means "delivers," and (-) means "takes back." For the second factor (+) means "checks," and

---

[1] Louis S. Cohen, "A Rationale in Working With Signed Number - Revisited," Arithmetic Teacher 13 (November 1966): 564-567.

[2] Peterson, "Fourteen Different Strategies for Multiplication of Integers or Why $(^-1)(^-1)=^+1$," p. 398.

(-) means "bills." The sign of the answer (simplified expression) is known by deciding whether the housewife will be richer (+) or poorer (-) as the result of the actions of the postman. For example, if the postman delivers three bills for twelve dollars, the housewife is poorer. Symbolically: $^{+}3 \times {}^{-}12 = {}^{-}36$. Conversely, if the postman finds that he has made an error in delivery, he takes back three bills for twelve dollars. Since he has taken back bills from the housewife, she is richer. Symbolically: $^{-}3 \times {}^{-}12 = {}^{+}36$.

This model relies on the students' willingness to accept the removal of debt as resulting in increased wealth for the housewife. It seems possible that some students would resist this interpretation.

A third example of a verbal model for multiplying negative numbers, this time using measuring, comes from Willerding.[1] As in the prior presentations, Willerding uses new language for the signs (+) and (-) depending upon their position in the expression. The first sign refers to minutes, either future, (+) or, past (-). The second sign refers to the direction of gasoline flow, either in (+), or out (-). The representation is of a gasoline pump which pumps gasoline in or out over time. For example, ($^{+}4 \times {}^{-}5$) refers to "four minutes from now, pumping *out* five gallons per minute, would result in a total of 20 gallons pumped *out*, or $^{-}20$. For two negatives in the expression, ($^-5 \times {}^-2$) we read five minutes ago, pumping out two gallons a minute, there were 10 gallons in the tank that are not there now, or $^+10$.

---

[1] Margaret F. Willerding, *Elementary Mathematics, Its Structure and Concepts* (New York: John Wiley and Sons, 1966), pp. 235-237.

In order for a student to be able to use this method he must be able to understand past time. This understanding should be available to most students but they would probably need some practice in applying their understanding. The mathematics committee at the Educational Development Center has pointed out that:

> Illustrative situations can be developed for motivating the multiplication of negative numbers (e.g. train traveling on a number line with certain speed in either direction; where is it at some time before T = 0?), but they usually rest on concepts like time and speed which are themselves difficult.[1]

Summary

To what extent are the concepts employed by Skemp, Cohen, and Willerding difficult? In these attempts to connect the multiplication of negative numbers to real life situations we find that the context of each method is from real life, but the operations within each of the contexts are not within real life. We could say, with Bruner, that these are attempts to structure mathematical relations such that they are within the grasp of the learner, but to what extent does the structure chosen for each method have merit? That is, Bruner has said that such merit depends upon the structure's power for simplifying information, generating new propositions, and being manipulable.[1] In this set of verbal methods we find that the stories which must be created to represent negative number multiplication tend to make the information more complex by adding both new vocabulary for the signs and factors and by requiring the student to cope with an artificial, contrived situation which, if ever it existed, would not be thought about in terms of negative numbers. In regard to manipulability and the generating of new propositions, it is difficult to imagine that students would be able to either manipulate or generate new propositions when the information they are working with is complex rather than simple. One of the requirements for manipulating and generating new statements is some degree of fluency with the material at hand. Inasmuch as the information has not been simplified by the context, it is doubtful that the necessary fluency could be developed.

---

[1] Education Development Center, "A Mathematics Curriculum for K-6," in Current Research in Elementary Mathematics, ed. Robert B. Ashlock and Wayne L. Herman, Jr. (New York: Macmillan, 1970), p. 293.

If one chooses to use a verbal model, or if the students have had sufficient experience to warrant the study of a verbal model, one should be aware of the differences among these models. Each employs a different context, but Skemp's and Willerding's models are similar in that they use measuring to evaluate a situation. Cohen's method is different in that it uses counting. Given the individual differences which students manifest, it seems well to avoid the repetition of a similar model when the need for remediation arrives. That is, if a student has shown confusion as a result of a particular method, a second presentation, while in the same mode, should probably be of a different type. Changing from a measuring requirement to a counting requirement or vice versa might well aid his understanding.

---

[1] Bruner, Toward a Theory of Instruction, p. 41. See chap. I p. 16, above.

Iconic Methods

One of the conclusions which the Education Development Center committee came to with regard to the multiplication of negative numbers was that they should concentrate their efforts into geometrical representations using segments or vectors.[1] The problem, as they noted, is to find ways of defining multiplication by means of natural constructions. These constructions, while they might turn out to be physical, would most likely be pictorial or iconic thus coming within the definition of the second of the three modes which we identified earlier.

Most iconic methods provide some sort of picture which will eventually be discarded in favor of mental manipulation of symbols. This is so in order that whatever understanding a student might achieve with a picture or a graph might be applied in the absence of that visual aid. However, many iconic representations do not have as their goal the learning of the multiplication of negative numbers, but rather the verification of one or another of the formal proofs. That is, the graphic representation is sometimes conceived with the mind that we know what the product is, how can we depict that product graphically? This is what the School Mathematics Study Group has done.[1] After establishing that the product of two negative numbers is positive with the use of a multiplication table, the authors state:

---

[1] Education Development Center, "A Mathematics Curriculum for K-6," pp. 293-295.

> ... if you are to keep the property of multiplication of counting numbers you must accept the following products: $(^-2) \cdot (^-2) = {}^+4$ and $(^-2) \cdot 3 = {}^-6$.[2]

A model is then introduced in the text which could be made into a picture. The following word problem is used:

> After losing 2 yards on each of 3 successive plays, a football team is on its own 30-yard line.
>
> (a) At what position was the team before these losses?
>
> (b) If you express a loss of 2 yards with the number $^-2$, the position of the team before the 3 losses is given by $30 - 3(^-2)$. Why do you subtract $3(^-2)$?[3]

In the situation and in question (a), no connection is established among the factors and what each represents. That is, "3 successive plays" does not carry negative meaning. Question (b) establishes the relationship as one of time but seems to rely on the prior knowledge of the table of products and on subtraction. The question directs the order of operations such that a next step looks like 30 - $^{-}6$. This brings about a concern for subtraction rather than the multiplication of two negatives which could also have been done. This example is one which represents the teaching method that presents a rule and then an example. In this case the example can be helpful since it can be visually illustrated. The understanding of the operations does not come from the picture when the presentation is in the order, rule-example. Instead the picture serves as a place of application for the rule. The example in this methodology is of the measurement type since it is concerned with moves on a continuum, the football field.

---

[1] School Mathematics Study Group, Studies in Mathematics, Volume 6, Number Systems (New Haven, Connecticut: Yale University, 1961), pp. 168-242.

[2] Ibid., p. 197.

[3] Ibid., p. 199.

Another example of an iconic representation which uses measurement to evaluate an expression or situation is the number line. This is one of the most popular of the iconic representations for negative numbers because it is widely used to introduce positive numbers and the four operations on them. And, with the practice provided in the positive numbers, the students are already familiar with the conventions attendant to its use. Coon[1] has developed a set of rules for representing the multiplication of two negative numbers on the number line. For the equation $(^{-}2) \times (^{-}3) = 6$ we are told to read the expression with a number line available. He states:

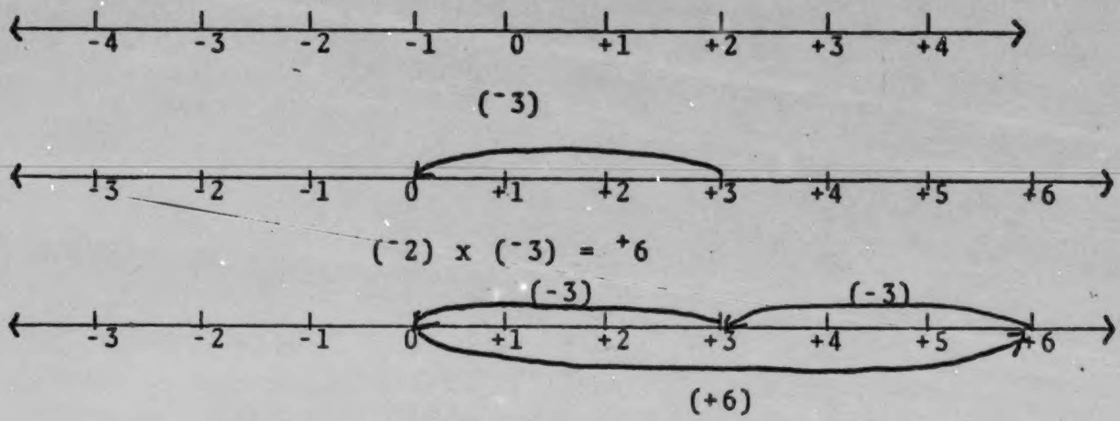
> Recognize that the first factor indicates the direction to draw vectors whose magnitude and direction are indicated by the second factor: (a) if the first factor is positive,

---

[1] Louis H. Coon, "Number Line Multiplication for Negative Numbers," Arithmetic Teacher 13 (March 1966): 215-217.

> the direction of moves agrees with that indicated by the second factor; (b) if the first factor is negative, the direction of moves is opposite to that indicated by the second factor.[1]

His representations of the expression are given below.[2]



It can be seen that this model relies heavily on the application of rules which are a priori to the representation. In other words, we know that a negative multiplied by a negative equals a positive, what system of rules for the number line will demonstrate this fact? The rules of negative numbers in multiplication and division are not generated by moves on a number line but they can be represented on it and this might be helpful to students. The requirement is that the student must accept the change in direction that takes place when the first factor is negative. The question is whether the student will accept this as a result of reason or as a result of need. It is doubtful that this representation will establish the "inner reasonableness" of the product.

---

[1] Ibid., p. 216.

[2] Ibid., Fig. 18, p. 216.

Another iconic-measure representation comes from Pratt,[1] who developed card packs to demonstrate the multiplication of negative numbers. These sets of cards operate like the old flip movies where each succeeding card shows the on-going action. In Pratt's methodology two packs are made with wagons or some other object shown at the top of the cards. Each card is made with the object displaced by one-eighth of an inch successively on each card. Pack one has the wagon on the left side so that the flipping of the cards gives the impression that the object is moving from left to right. This is defined as forward. Pack two begins with the object on the right-hand edge and a flip of this set gives the impression that the object is moving from right to left. This is defined as backwards. Pack one is associated with a positive first factor, pack two with a negative first factor. The sign of the second factor determines whether we should flip the cards forwards (+) or backwards(-). Simply organized, we see:

Expression	Pack to Use	Direction to Flip
(+) · (+)	1	Forwards
(+) · (-)	1	Backwards
(-) · (+)	2	Forwards
(-) · (-)	2	Backwards

By attending to the direction in which the object appears to move we can know the proper sign for the product of the expression. For example: (-) · (+), proper use of pack two will allow us to see the object moving backwards (right to left). By associating the (-) sign with "backwards" we have determined the sign for the product. To what extent this will aid a student's grasp of negative numbers cannot be known in the absence of experience. Here, as in the number line definition, it is not possible to abstract from the representation the logic of the operations. Again, this iconic representation serves only to verify other kinds of evidence regarding negative numbers.

---

[1] Edna M. Pratt, "A Teaching Aid for Signed Numbers," Arithmetic Teacher 13 (November 1966): 589-590.

Science Research Associates[1] has incorporated an iconic representation into their sixth level text. After presenting a formal definition by using the property of zero, positive and negative values are assigned to the quadrants of a traditional Cartesian graph. With the positive quadrants being the upper right and lower left and the negative quadrants being the upper left and lower right, the student is able to find the sign of the product by finding the quadrant in which the vectors of the factors intersect. He may then draw an array within that defined space to determine the quantity of the product. For example, figure 1 shows the graph and the result of the action described above when the expression to be evaluated is: ¯3 x ¯4.

---

[1] Science Research Associates, Elementary Mathematics Program, Book 6, Teacher Guide (Chicago: Science Research Associates, 1968), p. 333.

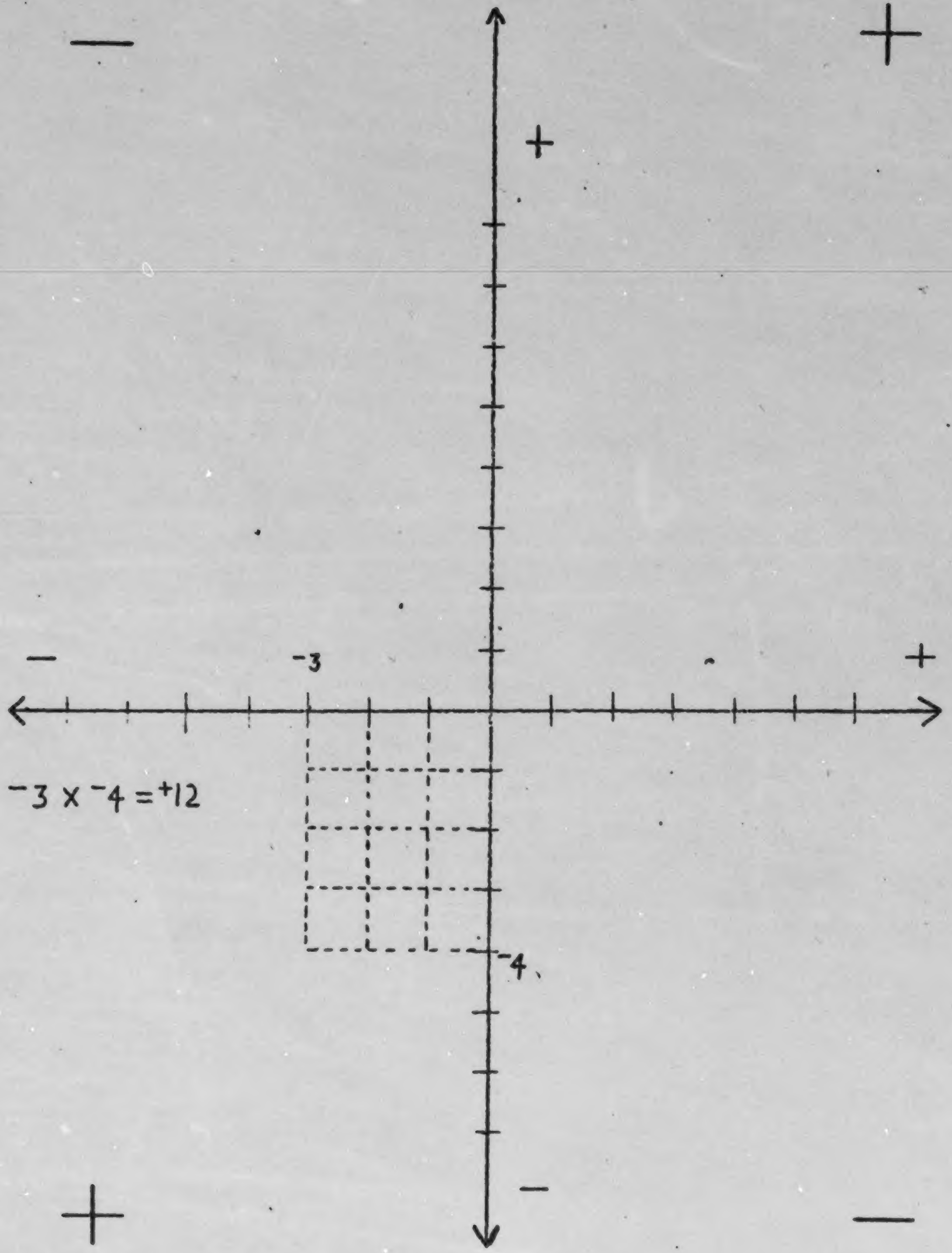

—
+
+
—
-3
+
$^{-}3 \times {}^{-}4 = {}^{+}12$
-4
+
—
—

Fig. 1. Graph, $^{-}3 \times {}^{-}4 = {}^{+}12$.

With this iconic-measure type representation a student can explore the products of negative multiplications. If he is familiar with the use of this kind of coordinate system he can define the rules for multiplying negative numbers. However, there is little within this method that serves to establish the reasonableness of the results obtained.

Another representation using a graph with a Cartesian coordinate system and "multiplier line" has been presented by Peterson.[1] A "multiplier line" may be either positive or negative. For either type of line we locate the point ($^+1$) on the X axis. If the line is to be positive we count up the indicated amount, for example $^+3$, count up three units. If the line is to be negative we count down the required number of units, for example $^-4$, count down four units. We then draw a line from the origin to the point we have marked. This line is then extended and becomes our "multiplier line." Figures 2 and 3 show a ($^+3$) and a ($^-4$) multiplier line.

---

[1] Peterson, "Fourteen Different Strategies for Multiplication of Integers or Why ($^-1$)($^-1$)=$^+1$," p. 400.

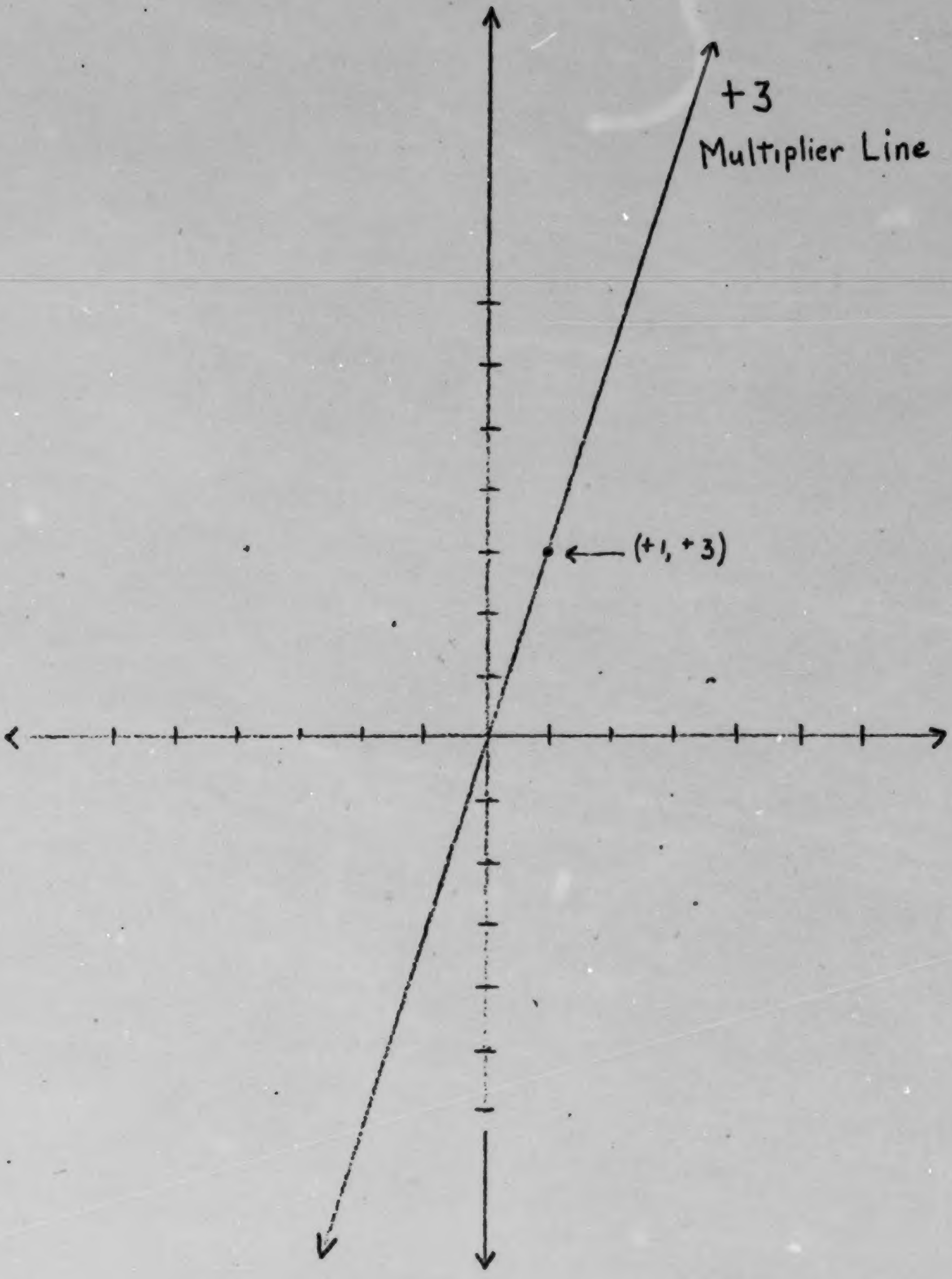


Fig. 2. $^+3$ Multiplier line.

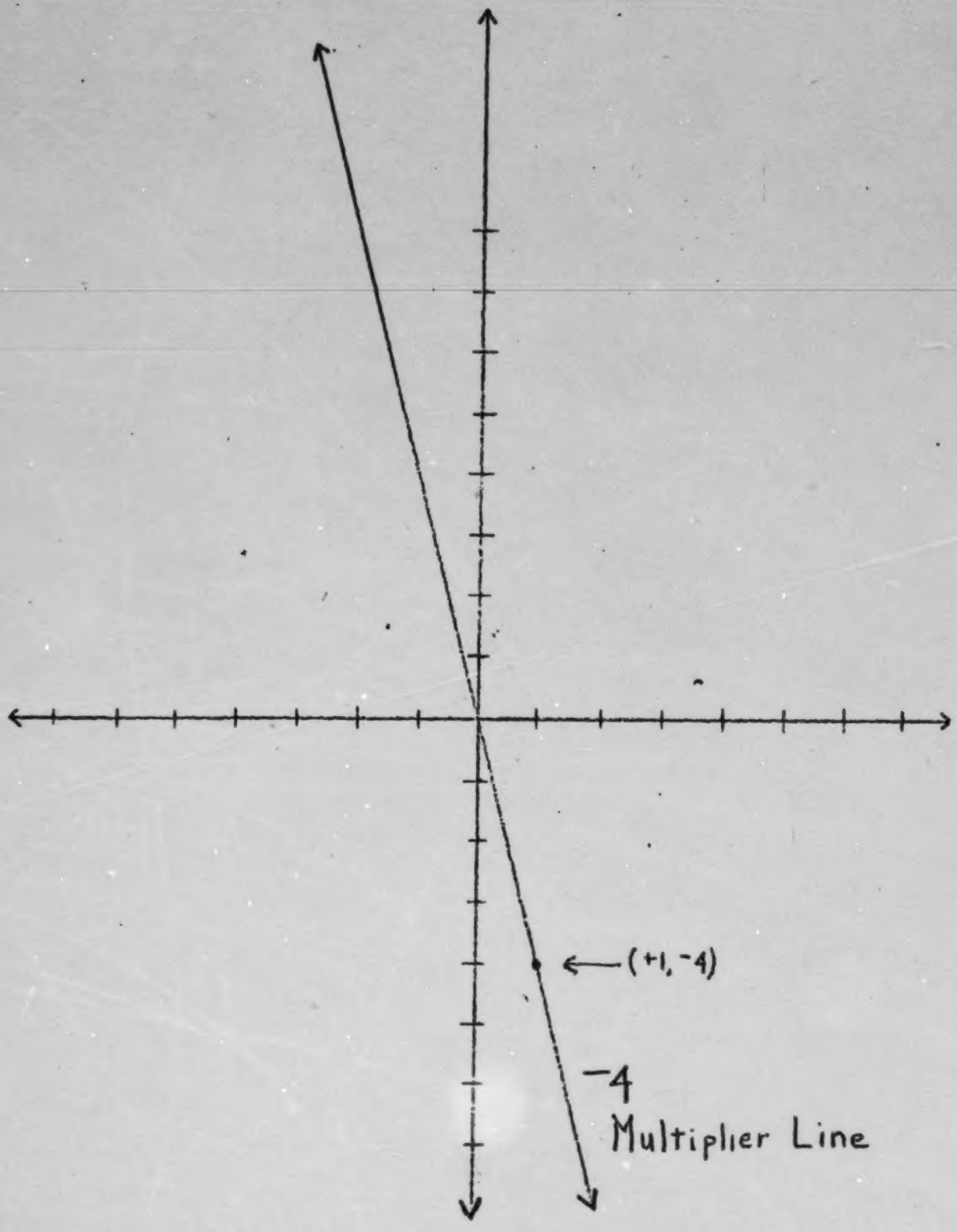


Fig. 3. ⁻4 Multiplier line.

In order to evaluate a product a line is drawn from the point on the X axis that represents the second factor in the expression to the point of intersection with the multiplier line which has been drawn from the first factor. Then a line is drawn parallel to the X axis toward the Y axis to see what the product and its sign is. Figure 4 shows the result of graphing ¯4 x ¯2 in this method. The multiplier line, first represented in figure 3 is drawn, then the (¯2) point on the axis is found. A line is then drawn from that point to the point of intersection with the multiplier line. Finally the line parallel to the X axis, directed toward the Y axis is drawn. The point at which this line intersects with the Y axis is the product of the multiplication.

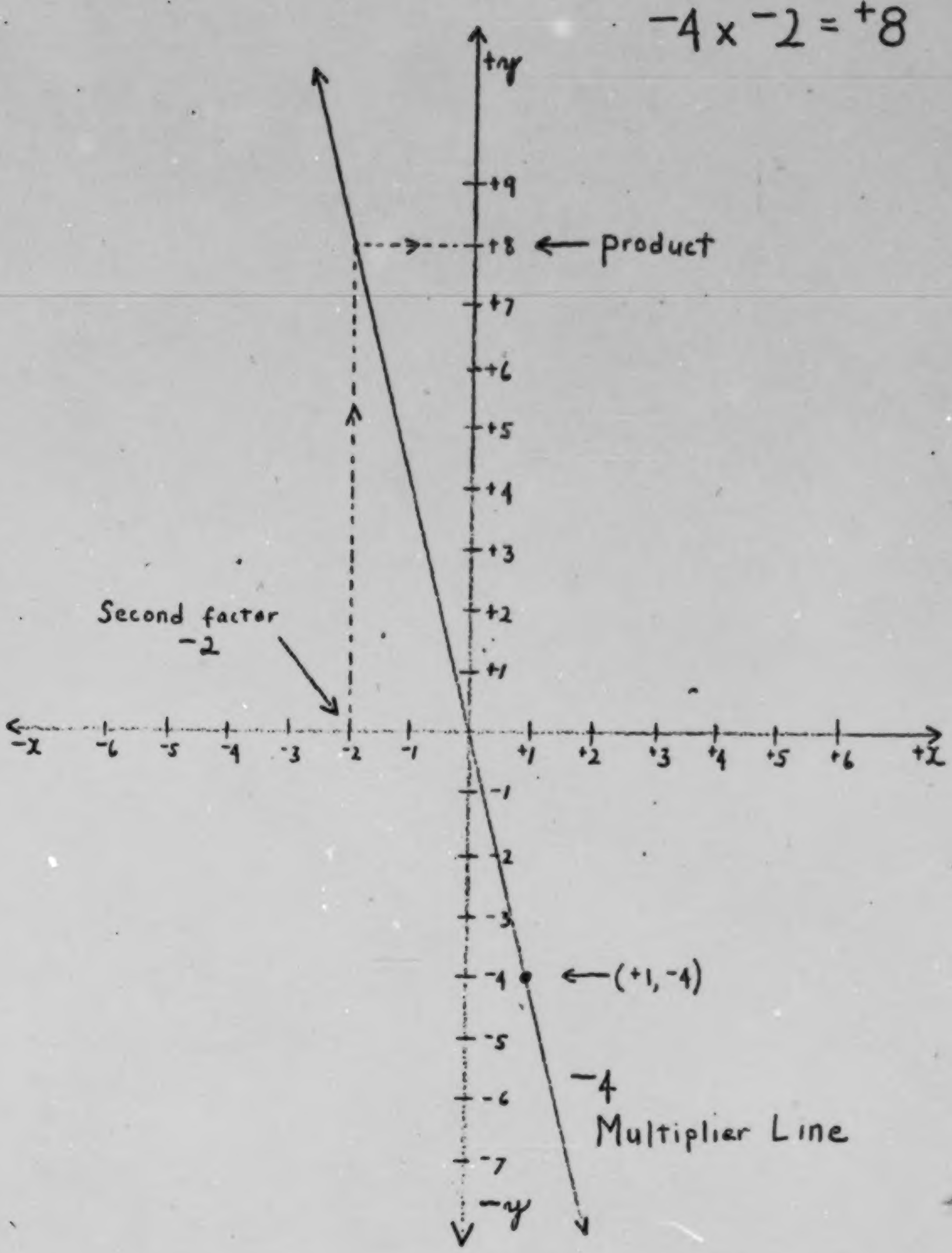


Fig. 4. Graph, $^{-}4x^{-}2=^{+}8$.

With this technique, any multiplier line may be drawn and a table of multiples of a number noted. The requirement for the student is again some familiarity with the coordinate system. However, the extent to which the determination of a product aids in establishing the "inner reasonableness" of a product remains open to question. It is likely an individual matter so that it would be wrong to conclude, absolutely, that this method, or any other method, does not establish the sought after reasonableness.

Cotter[1] has designed a method using the iconic mode but, in distinction to the above methods, he evaluates the product through a counting activity. He uses "charged particles" whose charge is noted by a (+) or (-) sign. The particles are operated on in a field (circle) where a (+) charge and a (-) charge have already been established as opposites which attract and cancel each other out. When this is the case, we draw a circle around the (+) over the (-), further exemplifying the neutral or zero state. The Cotter method does not require redefining or other changes to show three operations, addition, subtraction and multiplication. Consequently all three operations will be presented here. Starting with an empty field, we can place either (+) or (-) particles within it. Each act has an encoding depending upon what the field started with, what was done (added to, or taken from) and the result within the field. For example: empty field, 0; add four positives, 0 + $^{+}4$; result, 0 + $^{+}4$ = $^{+}4$. An example for subtraction is: zero field (any number of neutralized particles, i.e. (+) and (-) together), 0; subtract four negative particles, 0 - $^{-}4$; results in four positive particles, 0 - $^{-}4$ = $^{+}4$. By making sure that there is a sufficient number of neutralized particles any subtraction can be done in this way.

---

[1] Stanley Cotter, "Charged Particles: A Model for Teaching Operations with Directed Numbers," Arithmetic Teacher 16 (May 1969): 349-353.

Multiplication starts with a field of neutralized particles to which may be added N groups of positive or negative particles. For example: ($^{+}2$ x $^{+}3$ = $^{+}6$), add two groups of three positive particles. This results in a field of neutralized particles plus six positive particles.

> If the multiplier is negative, the interpretation can be made in terms of successive subtraction. Thus $^{-}2$ x $^{+}3$ is translated as subtract two groups of three positive particles each from the neutralized field.[1]

The multiplication of two negative numbers is done in much the same way except that we subtract negative particles.

> Here the subtraction of two groups of three negative particles each from the field results in liberating six positive particles from their neutral bonds, thereby changing the field from zero to positive six.[2]

This method seems to be well suited to representing some of the operations on negative numbers. The number of definitions necessary to operate with this representation is very low in comparison to others, and this means that a student has a greater opportunity to experiment with situations and their codings independently. Cotter does not develop a way for division to be represented, he comments that it should probably be delayed until multiplication has been formalized. It is not possible to say whether the logic of negative numbers can be abstracted from the use of this representation. It appears that, once the definition of neutrality is known, many of the logical relations of negative and positive numbers can be abstracted from activities with the picture. The question then relates to the notion of neutrality and whether this is firmly enough established by defining the neutral state as the combination of a positive and negative charge. Familiarity with physics may be a requirement that Cotter did not have to consider as his model was developed for college teaching. Albeit, it has the possibility of enhancing the understanding of many people.

---

[1] Ibid., p. 353.

[2] Ibid., p. 353.

Summary

In this last section which has examined some of the iconic representations for negative number multiplication we have seen that some of the methods are used for verification rather than development of rules. Each of them requires a great deal of defining before they can be used and most of the definitions are arbitrary rather than related to the nature of the representation. Further, all of them but Cotter's use measuring to evaluate the situation. Cotter's, which seems to be the most clear, uses counting. The drawbacks of the iconic methods seem then to be related to their use for verification, their need for definitions and their use of measurement for evaluation.

The iconic methods do have characteristics that make their structure have merit. That is, they do simplify the information to some extent; with the use of the football field, the graphs, and the charged particles new propositions could be generated by student conducted research. The flip cards, however, do not appear to be capable of generating new information, nor do they represent the information in such a way that the information itself is manipulable. Further, the manipulability of any of the methods is a matter of how well a student learns the techniques necessary to work with the methods.

This raises the question of the techniques themselves. The set of directions necessary for each method does not relate to the logic of the inter-relations of negative numbers. This is especially true of the graphs (both the array and the multiplier lines), the football field, and the flip cards. Cotter's method using charged particles is possibly an exception in that every activity required for operating with the method relates to the logic of the functioning of negative numbers. This point tends to recommend the Cotter method.

Physical Methods

The third type of representation for negative numbers is physical. One example of this type is described by

Hill[1] who uses all three types of representation (mental, iconic, physical) to establish negative number multiplication with a special set of elevators in a special building. We may imagine the elevators, or we may picture them with a vertical line, or we may build a physical model of them. The model has two types of elevators, down and up. For each type there are some that stop at every floor, some at every other floor, some at every third floor, etc. We can see that an up elevator that stops at every fourth floor would be encoded, ($^{+}4$). Similarly a down elevator that stops at every sixth floor would be written, ($^{-}6$). The other factor is the number of stops the elevator is to make and the sign of that factor is noted by whether the direction of the elevator is the same as its type (down or up) or different. The building has a ground floor (0) and many floors above (+) and below (-) ground.

Certain situations in the building generate movements by the elevators and the encoding of these movements and their result represent negative number multiplication. For example: If a person is on floor 24 and wants to come down he must call for a down elevator. But, in order for the down elevator to get to floor 24 it must go up, which direction is different (opposite) from its sign. So we choose an elevator which could stop at the proper floor. (In this case, a one, two, three, four, six, eight, or twelve.) Let us choose a down six, ($^{-}6$). This elevator must go up (the opposite direction from its type) four stops to arrive at floor 24. We can say then that a down six elevator must go in the opposite direction four stops to arrive at twenty four, $({}^{-}6 \times {}^{-}4 = {}^{+}24)$. One of the values of this model is that we can establish a similar expression for all of the other elevators capable of picking up the passenger, $({}^{-}8 \times {}^{-}3 = {}^{+}24$, ${}^{-}2 \times {}^{-}12 = {}^{+}24$, etc.).

---

[1] Warren H. Hill, Jr., "A Physical Model for Teaching Multiplication of Integers," Arithmetic Teacher 15 (October 1968): 525-528.

While Hill does not present plans for the construction of the physical model, it is easy to imagine the construction with the use of a long box (umbrella or golf club box) numbered zero at the middle and progressing up and down at four -- or six -- inch intervals from the middle. There should be at least two elevators whose type could be marked by an arrow and whose number of stops could be variable. These would be attached to strings which would be slipped through a hole at the top of the box with sufficient length for the elevators to reach the bottom of the building.

The physical representation described above uses measurement to evaluate a situation or expression. Hill has identified seven rules which must be known by students in order for them to understand the functioning of the elevators.[1] While these rules are not difficult they might get in the way for some students. If sufficient numbers of physical models were available, students could experiment with a variety of situations. They could generate a set of tables (stops for a particular elevator) and come to their own valid conclusions.

---

[1] Ibid., p. 526.

The danger in Hill's design is that only one model will be built and that too few examples will be presented before the model is disregarded. This is not a fault of the design, only of its application.

Another physical model, designed for use by individual students, has been developed by Fremont.[1] His design uses loops made from pipe cleaners. (These loops will be represented by parentheses.) He defines this loop ")" as "+" and the opposite, "(" as "-", so that (and) together () look like zero. Fremont uses this model for addition and subtraction only. He mentions the possibility that multiplication could be established as repeated addition, "... with the sign of the multiplier determining the final direction of the answer loops."[2] However, by applying the rules that Cotter[3] presented, an alternative method for multiplication is developed. For addition and subtraction Fremont uses the characteristics of zero as the identity element. Positive one may be represented as ")" or ") () () ()," that is, 1+0+0+0. We can now subtract from this number: $(^{+}1) - (^{+}4)$. In order to do this, we take away four positive loops: ~~)~~ (~~)~~ (~~)~~ (~~)~~ leaving, (((or three negative loops, $^{-}3$. The same act is valid for subtraction of a negative number: $(^{+}1) - (^{-}3)$. We begin with:) () () (); and we take away three negative loops:) $\cancel{(}$) $\cancel{(}$) $\cancel{(}$), leaving four positive loops, $^{+}4$. Fremont's idea for multiplication would be that the first factor would tell us the number and the direction of loops in a group and the second factor would tell us how many groups and whether to reverse the direction of the loops or not. A (-) sign for the second factor would mean "reverse the direction of the loops." For example: the expression, $^{-}3 \times {}^{-}4$ would require four groups of negative loops with three loops in each group. As the sign of the second factor is negative, we would be required to reverse the direction of all the loops, thereby giving a final count of twelve positive loops. If we apply Cotter's language and action to this model, we would begin with a neutral set, that is a set of positive and negative loops together, and use successive subtraction. With the same example, $^{-}3 \times {}^{-}4$, we would need at least 12 neutral loops from which we could subtract three groups of negative loops with four in each group. This would leave twelve positive loops. The two models are pictured below.

---

[1] Herbert Fremont, "Pipe Cleaners and Loops -- Discovering How to Add and Subtract Directed Numbers," Arithmetic Teacher 13 (November 1966): 568-572.

[2] Ibid., p. 572.

[3] Cotter, "Charged Particles: A Model for Teaching Operations with Directed Numbers," pp. 349-353. See chap. II, pp. 48-50, above.

Fremont:

((((((((((((

sign of second factor (-) means reverse

)))   )))   )))   )))

Fremont - Cotter:

() () () () () () () () () () () ()

sign of first factor means subtract

/() /() /() /() /() /() /() /() /() /() /() /()

The major difference among this method and the others in this chapter is that the manipulation of these loops requires only a minimum of conventions, and the manipulation also establishes rather than verifies the results of negative number operations. This serves to make the tool a useful one in that it might be possible for a student to abstract the logic of negative numbers from the manipulation of the material. However, certain students might not be able to accept the arbitrary assignment of positive and negative values to loops.

The last example of a physical model uses Cuisenaire rods. Kunz[1] developed a method which uses the rods, a defined field, and prior knowledge of factoring in negative numbers. The rods will be described in chapter three. The defined field is simply a sheet of paper divided in half by a line in order to have a positive, upper half and a negative, lower half. There is also a space on the field for defining the unit, inasmuch as any rod may serve as the unit. The prior knowledge required is that a negative number ($^-5$) is equivalent to 5 x $^-1$. Negative one is then established as a factor which causes a 180 degree rotation of the position of anything in the field. For example, if a rod is in the positive field, the rotation takes it to the negative field. If we want to multiply $^{-}5$ x $^{-}3$ we place 3 in the negative area and factor the negative five to get, 5 x $^{-}1$. We then use the rotation rule causing the three in the negative space to move to the positive space. We can now cross the three with the positive five giving the result, $^{+}15$. Kunz recommends the use of matrices (tables) for the students to record what they find in applying this rule. This is an example of a physical model which uses measurement to evaluate the product and a rule to find the sign of the product. Inasmuch as the model uses a predefined rule to aid in the result of the manipulation, it is not possible for a student to experience the logic of negative numbers with the material itself. It is basically a physical model for verification though it may be used before the presentation of a proof. The use of the rule for negative one, while serving to free the students for exploration, does not serve to establish the reasonableness of the result found through the manipulation.

---

[1] Interview with John Kunz, Cuisenaire Company of America, New Rochelle, New York, 17 October 1974.

Summary

The physical methods do simplify information to some extent; the elevators require rules, but once they are known the information is well organized; the loops are especially simple inasmuch as few rules are necessary; and the rods and field method, when the rule is in hand, also serve to simplify the information. All of the methods are capable of allowing a student to generate a great deal of data regarding various multiplications. The methods are, however, limited to integers and three operations of addition, subtraction, and multiplication. The primary value of the physical methods is that they are manipulable by the student. We know from chapter one that it is action that produces the possibility of abstraction. And, in the respect that they do admit to action, the physical methods present new possibilities for preparation for the logic of negative numbers. That is, Piaget and Dienes both talked about play and the concurrent action as preparation for the structures of logic and mathematics, it seems reasonable to look at these manipulatives, designed for negative numbers, to see what their value is regarding play. Here we encounter difficulties because we find that none of the physical manipulatives admits to play. We can play with them. But, in that playing, we will probably change them in such a way that they no longer represent negative numbers. That is, the materials are not built such that they continue to represent negative numbers in any configuration. The elevators need their rubrics, the loops need their direction, and the rods need their field and rule. Thus, while action is available and necessary it is not preceded by play because the play will probably not prepare the student for the action. Simply stated, one of the major purposes of manipulative materials is not possible to pursue with these designs.

Summary

This concludes our survey of the representations for negative numbers and their use in the operation of multiplication especially. The purposes of the chapter have been to identify, classify, and compare current practices. Our comparisons included comments upon the extent to which current practice has been informed by the theory presented in chapter one. Our classification scheme was derived from two sources: Ashlock and West,[1] and Bruner,[2] and we found six different ways of representing negative numbers. We began our survey with a glance at methods for proof of negative number multiplication and continued by noting various ways of working with signs and their contexts. We then examined examples of verbal representations, continued with iconic representations, and concluded with physical representations. The methods we looked at can be classified as follows:

Author	Representation	Mode	Type	Page
Skemp	staircase	verbal	measure	31
Cohen	postman stories	verbal	count	32
Willerding	gasoline pump	verbal	measure	33
SMSG	football field	iconic	measure	37
Coon	number line	iconic	measure	38
Pratt	flip cards	iconic	measure	40
S.R.A.	signed quadrants	iconic	measure	41
Peterson	multiplier lines	iconic	measure	43
Cotter	charged particles	iconic	count	48

---

[1] Ashlock and West, "Physical Representations for Signed-Number Operations," p. 549. See chap II, p. 26, above.

[2] Bruner, Toward a Theory of Instruction, pp. 10-11. See chap. II, p. 26, above.

Author	Representation	Mode	Type	Page
Hill	elevators	physical	measure	52
Fremont	loops	physical	count	54
Kunz	rods-field	physical	measure	56

The general purpose of the verbal representations is to establish a mental model with which to think through the dynamics of negative number multiplication. We found that these required the attaching of various words to signs and factors in order to make the model fit the symbolic representation. The Education Development Center committee[1] reminded us that some of the concepts that are required for understanding these situations are difficult and possibly a geometric, graphic representation would be more useful. Other questions related to whether the methods were capable of establishing what the Cambridge Conference[2] termed the "inner-reasonableness" of the product of two negative numbers.

All of the comments we have made above are relevant to the criteria of simplifying information and the consequent abilities to generate new propositions and manipulate the information. Piaget and Dienes stress the use of physical materials so that their comments are not totally relevant here except for their opinions regarding abstraction. Both theorists see the learner abstracting from his action, in this case a purely mental action, but the learner should be focussing on the actions rather than on the material, in this case, the context. Are the contexts we found in the verbal methods too complex for a learner to be able to concentrate only on the action? We think that they probably are.

---

[1] Education Development Center, "A Mathematics Curriculum for K-6," p. 293. See chap. II, p. 34, above.

[2] Cambridge Conference on School Mathematics, "Curriculum for the Elementary School (K-6)," p. 15. See chap. II, p. 28, above.

We found that the majority of the iconic representations use measurement and, more importantly, that they seem to be intended for verification of symbolic manipulation rather than precursors of symbolic manipulation. The exceptions we found were Peterson's "multiplier lines"[1] and Cotter's "charged particles,"[2] both of which seemed to be capable of establishing the rules of negative number multiplication. We noted that the Peterson method had no attributes which would serve to establish the "inner reasonableness" while Cotter's method had that possibility where neutrality was understood.

Within the use of the iconic representations there is a possibility of play. However, such play would probably serve the purpose of learning to use the representation itself rather than the logic of negative numbers. The only action in this set of methods which relates to negative numbers themselves is the action called for by Cotter when the positive and negative charges are joined by a circle and made neutral.

---

[1] Peterson, "Fourteen Different Strategies for Multiplication of Integers or Why $(^{-}1)(^{-}1)={}^{+}1$," p. 400. See chap. II, p. 43, above.

[2] Cotter, "Charged Particles: A Model for Teaching Operations with Directed Numbers," pp. 349-353. See chap. II, p. 48, above.

In the physical methods we found in Hill's "elevators"[1] and Fremont's "loops"[2] the possibility that students could manipulate the material. We were not satisfied that such manipulation would establish the logic of negative numbers but the possibility is there. The difficulty with the elevators was caused by the seven rules and the construction of sufficient numbers of models to permit student exploration. The loops seemed the most likely to be able to establish the "inner reasonableness." The question with regard to the loops was whether students could work with the arbitrary definition necessary. The Kunz[3] method is dependent upon pre-established rules so that, while experience could be provided, the use of an arbitrary rule probably serves to hinder the possibility of establishing the "inner reasonableness."

With regard to play and actions we noted that the Hill method (elevators) probably does not permit play but that action with it is possible. The Fremont method (loops) permits both play and action but there is a difficulty in the play activity inasmuch as the loops are representative of negative numbers only in certain positions in space. During play they are likely to be rotated and otherwise moved around and these actions will not be in preparation for the later work. The rods admit to play, but in conjunction with the field require the use of a rule so the problem of play as preparation continues to obtain.

---

[1] Hill, "A Physical Model for Teaching Multiplication of Integers," p. 526. See chap. II, p. 52, above.

[2] Fremont, "Pipe Cleaners and Loops -- Discovering How to Add and Subtract Directed Numbers," p. 572. See chap. II, p. 54, above.

[3] Interview with John Kunz, Cuisenaire Company of America, New Rochelle, New York, 17 October 1974. See chap. II, p. 56, above.

Abstraction, whether defined by Piaget or Dienes, is related to action; and as noted above, action is available with the physical methods. We can have the many experiences that Dienes[1] calls for, and to some extent we can create a class. But all of this is available only when we are able to disregard actual action and begin to perform virtual action as defined by Servais.[2] Are the physical representations simple enough to allow the disregarding of the materials themselves? The Fremont model seems to be the only one where this is possible. However, the unitary nature of it limits the variability available on a perceptual and a mathematical level.

We can conclude that what is necessary is a physical manipulative to represent negative numbers which maintains its essence as a negative material through any and all translations in space. When this is the case, play and the concurrent action will relate to the logic of negative numbers and the consequent abstraction will follow naturally. In other words, we need a material that is so structured that students, in any activity, will be experiencing the logic of negative quantities as well as the results of operations on those quantities.

---

[1] Dienes, "Some Basic Processes Involved in Mathematics Learning," pp. 21-34. See chap. I, p. 12, above.

[2] Servais, "The Significance of Concrete Materials in the Teaching of Mathematics," pp. 207-208. See chap. I, p. 18, above.

Before proceeding to a presentation of what such a material might look like, it is necessary to examine the concrete manipulative which provides the base for extension into the negative numbers. We noted earlier that the material to be extended is the Cuisenaire rods. Consequently, an examination of the theory and research relating to the rods is in order. This will provide a knowledge of the rods, the possibilities for their use, and, to the extent that the research is valid, the results of their use. After this preparation, chapter four will present the design that fulfills the need outlined above.

Chapter III

THE CUISENAIRE RODS: THEORY AND RESEARCH

Introduction

In 1914 James McLellan and John Dewey talked about the design of a manipulative and described its features thus:

> ... since exact ideas of numbers can arise only from exact measurements, and since ten is the base of our system of notation, the metric system can be used with great advantage. The cubic centimetre (block of wood) may be taken as a primary unit of measure; a rectangular prism (a decimetre in length), equal to ten of these units, will be the 10-unit, and, ten of these the 100-unit. The units may be of different colours, and the units of the decimetre alternately black and white.[1]

The dimensions outlined in the above plan are precisely those of the Cuisenaire rods. George Cuisenaire, the designer of the rods may or may not have seen this design before he fashioned his set of sticks. The Cuisenaire design for manipulatives has rods of from one to ten centimeters in length with the color of each length remaining constant. That is, all of the pieces of one (1) centimeter are white; two (2) centimeters, red; three (3) centimeters, light green; four (4) centimeters, purple; five (5) centimeters, yellow; six (6) centimeters, dark green; seven (7) centimeters, black; eight (8) centimeters, brown; nine (9) centimeters, blue; ten (10) centimeters (a decimeter), orange. The color assigned to each length is not arbitrary. There are families of lengths in which the colors are related. There is the red family:

---

[1] James A. McLellan and John Dewey, The Psychology of Number (New York: D. Appleton and Company, 1914), p. 182.

red, purple, brown, being two (2) centimeters, four (4) centimeters and eight (8) centimeters. There is the green family: light green, dark green, blue, being three (3) centimeters, six (6) centimeters and nine (9) centimeters respectively and there is the yellow family: yellow, orange, being five (5) centimeters and ten (10) centimeters. The white rod and the black rod have no related colors since the white is the unit of all the rods and black is the only prime length for which there are no multiples within the set of ten rods.

When the association of color with length is consistent, the user can know that when a yellow rod does not serve his purposes in a manipulative situation, then no yellow rod will work since they are all of the same length. Whether the color families are of educational value is questionable and is probably a matter deserving of some research.

One educational benefit in the use of the rods is that they allow, as do other manipulatives, "... the pupil to gather his mental experience at his own pace without the authority of the teacher."[1]

John Trivett points out that when the rods are available for study, each student may be encouraged to determine for himself precisely what he sees in his constructions. Trivett says that the student's dialogue with the rods helps to give a deep understanding of mathematical structures. Therefore, given that the dialogue with the material is student-directed, the approach may be truly individualized.[1]

---

[1] Servais, "The Significance of Concrete Materials in the Teaching of Mathematics," p. 207.

Theory

Caleb Gattegno, the man who is credited with making the Cuisenaire rods well known, especially in English speaking countries, has written extensively on both the theory and the practice of using the Cuisenaire rods. Gattegno, as does Trivett, speaks of

> ... the immediate dialogue into which the child enters with the material in order to find mathematical structures in it which permit a rapid and a correct algebraic dynamism.[2]

In other words, the rods permit effective action and, by their nature, direct that action into algebraic (i.e. addition, subtraction, multiplication, division, extraction of roots) situations. And, with the color-length consistency the child does not have to perform every action to know what its outcome would be

> In fact, the characteristic feature of mathematics is to work in the virtual, to perform only those actions that are indispensable, combining them mentally to transform them and save time, and to leave the others aside. This the child will do from the outset of his own accord and will do very well.[3]

---

[1] John V. Trivett, "The Cuisenaire Rods -- Numbers in Color," Mathematics Teaching, No. 18 (1962), p. 40.

[2] Caleb Gattegno, For The Teaching of Mathematics, 3 vols. (Reading, England: Educational Explorers, 1963), 3:15.

[3] Ibid., 3:16.

This distinction between the virtual and the actual regarding actions is most important to Gattegno. He reiterates the mathematician's mental attitude in many of his essays.[1] For example:

> To put it briefly, the essence of the mental attitude which characterises the mathematician is the substitution of the virtual for the actual, the performing of virtual actions which are envisaged as being real or realisable, once there is no reason to suppose that they might not become actual.[2]

He continues by describing his theory of how a child, working with the rods, begins to develop and operate with a mental attitude equivalent to the mental attitude of the mathematician. During the early period with the rods, the period called "qualitative arithmetic," the child, through experience, is provided with the "... mental structures and dynamism which will form the basis of numerical arithmetic."[3] In free-play activities Gattegno sees certain awarenesses developing in the child, such as the awareness of "... what his action can, and cannot achieve."[4] That is, by discovering the various positions in which the rods can be placed, and by discovering that certain spaces can be filled by some rods and not others the learner will be able to reduce the number of actions, avoiding useless ones and eventually "... come to use his vision to form criteria which will precede action."[1] We see in the above description the transition from the need for actual actions, during a trial and error phase, to the virtual actions available to the child when he uses his visual criteria: when decisions are made about the rods without actually placing them. Gattegno points out that this activity is not yet directed towards mastery as the actions of the child, whether actual or virtual, are not selected with a view toward forming a particular structure. One such structure might be a train, (rods placed end to end). For example: a set of three particular rods could be chosen such that two of them, when placed end to end in a train, form the length of the third. When train-building begins, so also begins the movement toward mastery of actions. The train red and light green forms a length equivalent to yellow. If one of the rods in the train, either red or light green is removed, trial and error activity is soon replaced by attention to color (and therefore length) in finding the missing one. Again the shift in actions has been from the actual to the virtual. If, as Gattegno says, the child is really attending to color then he knows in a virtual sense that that color is precisely what will fill the created gap. His knowing is not dependent on actual action. Gattegno describes some of the mathematics available with this three rod assembly: A+B=C; B+A=C; C-A=B; C-B=A. Transpositions of the signs can create many more, (e.g. C=A+B). Gattegno goes on to show that the same kind of operations, with additional possibilities, are available with a set of four rods, two of which placed end to end in a train form a length equal to the other two placed end to end in a train, (e.g. A+B=C+D). He notes that when certain rules are applied to train-making, the nature of odd and even numbers, prime and composite numbers, and divisibility can be studied qualitatively. For the study of odd and even numbers, there will be a requirement that all trains be made with a red rod and, if necessary, one white rod at the end of the train. By measuring all ten of the rods in the set in this way one may see the pattern that alternate rods can be measured by red alone, without the white. At the same time we will see that, beginning with the first rod (the white one), the white rod must be added to the end of every other train in order to form the given length. One result of this study could be a chart:

---

[1] Ibid., 3:1-128.

[2] Ibid., 3:43.

[3] Ibid., 3:47.

[4] Ibid., 3:48.

---

[1] Ibid.

Odd	Even
white=white	red=red
light green=red+white	purple=red+red
yellow=red+red+white	dark green=red+red+red
black=red+red+red+white	brown=red+red+red+red
blue=red+red+red+red+white	orange=red+red+red+red+red

For the study of prime and composite numbers there will be rules that 1) eliminate white trains, and 2) require that all trains be made of the same color rod. For example, if one chooses to study the dark green length, he can see that he can make a red train equivalent to that length, as well as a light green train equivalent to that length. If he had chosen the black length he would have been unable to find a train of one color that would form the same length. Therefore, he could say that the dark green length is composite, and the black length is prime.

Prime	Composite
white	
red	
light green	
	purple = red+red
yellow	
	dark green = red+red+red = light green+light green
black	
	brown = purple+purple = red+red+red+red
	blue = light green+light green+light green
	orange = red+red+red+red+red = yellow+yellow

Gattegno states that: "... children who have become aware of these structures are ready to go straight into the study of arithmetic proper."[1] By recognizing two features of the nature of the rods, the child will be able to engage in the set of virtual actions that characterize mathematical thinking. The features necessary are: 1) that the rods, when white is the unit, are equivalent to the succession of numbers 1 through 10 and that these number names correspond to the making of two rod trains formed by continuing to add one white rod to each succeeding length:

---

[1] Ibid., 3:49.

Rods	Number Names
white	1
white + white = red	1+1=2
white + red = light green	1+2=3
white + light green = purple	1+3=4
white + purple = yellow	1+4=5
white + yellow = dark green	1+5=6
white + dark green = black	1+6=7
white + black = brown	1+7=8
white + brown = blue	1+8=9
white + blue = orange	1+9=10

and, 2) that whatever length is chosen for study, a set of decompositions of that length may be interpreted in terms of mathematical activities. That is, starting with a given length it is possible to find all of the ways to make trains of the same length using different sets of rules resulting in various types of trains. In these trains one can see additions, subtractions, multiplications, divisions, divisions with remainders, factors, and fractions. For clarification:

Type of Decomposition	Rule	Mathematics
partitions	any number of rods of any color placed end to end to form a length	addition, subtraction
complements	two rods of any color placed end to end to form a length	addition, subtraction
factors*	any number of rods of one color placed end to end to form a length	successive addition, subtraction, multiplication, division, factors, fractions

*In this study, division with remainder is possible if we amend the rule to allow trains of one color plus whatever else is needed to form the length.

Gattegno summarizes the result of these activities and awarenesses with the following:

> Qualitative arithmetic, then provides the mathematical mental structures, whereas the decompositions provide the numerical habits, the experiences which are ordinarily classed as arithmetical. From the point of view of truly mathematical activity it matters little whether the result of an addition is 7 or 83; what matters is that the child shall be aware that if he makes an addition he can unmake it by the appropriate subtraction, that the evocation of factors entails that of the result of their multiplication, and so on. In fact, although what is always stressed is the acquisition of details, of "numerical facts," what forms arithmetical ability is mastery of operations. With the Cuisenaire materials it is this mastery that is acquired, naturally and from the start.[1]

---

[1] Ibid.

Summary

Gattegno talks of preparation through physical manipulation, but he expects a much more rapid development of awareness than does Piaget, Dienes, or Bruner. Using the above postulates, Gattegno developed a curriculum for elementary mathematics called "The Cuisenaire-Gattegno Method."[1] This method stresses operations as opposed to the "details" which he criticizes in the last paragraph. It may be said that what Gattegno sees as being possible with the rods has been organized so that others may approach the possibilities he sees. Madeleine Goutard has demonstrated with children some of the points which Gattegno makes and has shown that, given a thoroughly prepared teacher, young children can do all of the mathematics outlined by Gattegno.[2]

The theoretical possibilities, both psychological and pedagogical, that have been outlined in this section support inclusion of Cuisenaire rods in the set of concrete manipulatives used in the classroom.

In the next section some of the research on Cuisenaire rods will be examined with the possibility of learning more about their usefulness and their value in the classroom.

---

[1] Caleb Gattegno, Mathematics With Numbers in Color, 4 vols. (Mount Vernon, New York: Cuisenaire Company of America, 1963).

[2] Madeleine Goutard, Mathematics and Children (Reading, England: Educational Explorers, 1964), ch. 3.

__________, Experiences With Numbers in Color (New Rochelle, New York: Cuisenaire Company of America, 1974), ch. 10.

Research

The preceding section has dealt with theory underlying the use of Cuisenaire rods. This section will present a survey of the research on Cuisenaire rods. For the most part, the research has been comparative. That is, Cuisenaire rods have been compared to traditional methods of teaching, or to other methods specifying the use of manipulatives.

Introduction

The major portion of the research to be reported in this section has examined the Cuisenaire-Gattegno method. A close look at this method reveals inconsistencies between the content of it and the content of the traditional textbook course of study. These inconsistencies are exemplified by Gattegno's statement regarding details; that children should be required to study operations rather than details. One example of this orientation is the omission from the Cuisenaire-Gattegno method of certain practice techniques present in textbooks. One of the important omissions in the method is the word-problem or problem solving portion of the curriculum. But the purpose here is neither to justify nor indict the Cuisenaire-Gattegno conception of the arithmetic curriculum. Rather, it is simply to point out that the research being reported here does not, for the most part, attempt to evaluate the rods as a manipulative aid for enrichment of a standard curriculum organization. Nevertheless, this is by far the most prevalent way in which the rods are used today. Two major publishers have incorporated the use of the rods into their regular mathematics courses, Addison-Wesley's Investigating School Mathematics[1] and the I.M.S. program published by the Center for Individualized Instructional Systems.[2] Also, the Cuisenaire Company reports that it has sold well over 50,000 classroom sets of rods. It is likely that most of this equipment is being used as enrichment material either in the regular classroom or in the math lab. Next steps for research should include some experimental studies to determine the usefulness of the rods in this context.

One researcher, John D. Williams,[3] has outlined some of the problems encountered when comparing teaching methods. Inasmuch as his outline has been generated in response to problems he faced when evaluating the Cuisenaire-Gattegno method and a method using the Dienes multi-based arithmetic blocks it is appropriate to use his report to help provide insight into the validity and usefulness of the research conducted thus far. He identifies the following problems:

1) Equating schools and pupils -- in order for a comparison to be useful, pupil ability and school setting must be equated.

---

[1] Robert E. Eicholz, Phares G. O'Daffer, and Charles R. Fleenor, Investigating School Mathematics (Reading, Massachusetts: Addison-Wesley, Inc., 1973).

[2] Frank C. Emmerling, Developer, Individualized Mathematics System (Durham, North Carolina: Center for Individualized Instructional Systems, National Laboratory for Higher Education, 1971).

[3] John D. Williams, "Some Problems Involved in the Experimental Comparison of Teaching Methods," in Mathematical Learning, ed. Leonard Feldman (New York: Gordon and Breach, 1969), pp. 157-191.

2) Teacher ability -- the relative abilities of different teachers are bound to affect results. Williams used the same teachers to teach different courses.

3) Teacher selection -- where teachers choose to participate, enthusiasm has its effects. When the program has been adopted by a principal without reference to the teachers, one might encounter "... a heightened degree of dissatisfaction and bewilderment among teachers ..."[1]

4) Teacher experience -- when the teachers who use the methods being compared are at different levels of experience, results may not be valid. Also, the amount of experience with a particular method seems to be important.

5) Teacher and Class constancy -- this heading refers to the problems of transfer of students and retirement or transfer of teachers.

6) Conditions -- control and experimental -- The conditions within each setting should be as normal and as equal as possible.

7) Assessment -- as Williams notes:

> One of the problems in the comparison of two teaching methods is that of constructing testing instruments that will favour neither, and yet register the special merits of both. This problem seems to have defeated many of

---

[1] Ibid., p. 168.

> those who have tried to assess structural methods, for, in many cases, conventional arithmetic tests have been used in the assessment, and such tests -- insensitive as they are, to the kind of outcomes at which structural methods aim -- could not have been expected to reveal the particular merits of these methods.[1]

Another facet of the assessment problem is the difficulty of measuring understanding. Williams points out that there are many definitions for the term and that it is difficult to construct a test that measures understanding in that the requirements for success on the test might be met through rote recall. A peripheral problem, related to assessment, should also be mentioned. The peculiarities of the Cuisenaire-Gattegno method have already been noted, but it is necessary to note further that structured materials, even when used for enrichment, will most likely alter the sequence and the scope of any mathematics course. For example, some amount of time should be given over to exploring the materials and this is time away from the predefined scope and sequence. Often, researchers and teachers do not reflect in their objectives and plans either the time requirements, or the consequent changes in scope and sequence that the structural materials might cause. Therefore, as the overall goals have not changed, the traditional test still has the appearance of validity. However, a thorough review of actual practices might show the traditional tests to be invalid.

---

[1] Ibid., pp. 180-181.

8) Experimental stimulation -- The effect of the experimental situation on increasing the efforts of all concerned, student, teachers, and administrators is difficult to measure. It seems likely, according to Williams, that some degree of achievement can be discounted as being the result of added interest on the part of the participants.

The above eight points provide a framework with which to examine the research pertaining to the Cuisenaire-Gattegno method and to Cuisenaire rods themselves.

Survey

In 1962, Gagnon[1] *et al.* attempted to determine the relative effectiveness of the Cuisenaire method and a traditional method by constructing two tests each for grades one, two, three, four, and seven. One test was described as traditional and the other was built around possibilities pursuant to the use of the Cuisenaire method. No attempt was reported to assign the groups on a random basis or to make them equivalent in any other way and no tests of reliability were reported in the study. The results appear to show that there is no significant difference between Cuisenaire and traditional classes on the traditional test. The Cuisenaire test produced marked raw score differences in performance favoring the Cuisenaire group but no statistical tests were presented. However, the Cuisenaire test given at the various grade levels had no more than one question relating specifically to the use of rods and two of the tests had no questions of that type. Nevertheless, without equating groups, and without proper statistical treatment little may be said about the results. Root,[1] (1962) reported a comparison of scores on a traditional test among grades one, two, and three. The comparisons were among children who had taken the test in 1958, before the Cuisenaire method was adopted, and children who had been taught with the method. These children, depending on grade level, had had up to three years instruction in the method. The data reported are comparisons of percentages of children at or above two specific percentile points being: above the 90th and below the 50th percentiles. Also, the class median percentile was reported. While the numbers clearly favor the Cuisenaire group, it is not possible to know how useful these results are as there was no evidence that the groups were equivalent, and there was no information reported regarding the reliability of the measures or the statistical significance of the differences. The Root study does not attempt to justify itself as a research document. It is a simple reporting of scores on a before and after basis using the same standardized test.

---

[1] Roger Gagnon *et al.*, "Etude Comparative -- Méthode Cuisenaire -- Méthode traditionelle," trans. John Trivett. Montreal, 1964. (Mimeographed.)

---

[1] Lawrence T. Root, "One School's Experience with Cuisenaire-Gattegno Mathematics Instruction," New York State Mathematics Journal 13 (January 1963): 24-27.

Karatzinas,[1] (1958) reports no statistically significant differences on addition and subtraction tests among students using the Cuisenaire method and a control group on Schonell's Diagnostic Arithmetic Test except that the Cuisenaire group was able to take the multiplication, division, and fractions parts of the exam whereas the control group had not had sufficient exposure to these operations in the opinion of the investigators, to warrant giving them those parts of the test. The subjects studied were seventy-eight second grade boys and girls who had been matched by Thurstone's Primary Mental Abilities Test. Forty children were in the experimental group, thirty-eight in the control group. Although the sample is small, the results of this study point to the possibility that more topics may be presented to young children than simply addition and subtraction. This possibility is supported by Sweeney[2] (1964) whose research will be discussed later in this chapter. The fact that the Cuisenaire children were able to take the extra parts of the examination, and that they took the extra parts, does not mean that the test employed by Karatzinas was able to register the "special merits" of the Cuisenaire method. The only thing that was done in response to the differences in the Cuisenaire method's scope and sequence was to extend the test. Another report, Ellis[1] (1964), deals with the longitudinal question of whether any benefits from the Cuisenaire method are maintained over time, in this case, five years. However, the question was dealt with by using the regular examination so that again, the benefits that might result from the Cuisenaire method were not fully examined. There were no significant differences between Cuisenaire and control groups but the raw scores of the Cuisenaire group were consistently higher than those of the control groups. This difference increased as the number of years of instruction in the method increased. Ellis also notes that there was a tendency for scores on the "reasoning" section to go down with length of Cuisenaire instruction although these tendencies were not statistically significant. This might be explained by the reverse of the practice effect. As the Cuisenaire method employs very little "reasoning," that is, problem solving practice, scores would suffer by comparison as more and more practice was available to the controls.

---

[1] D. Karatzinas and T. Renshaw, "Testing the Effectiveness of the Cuisenaire Material," Scottish Educational Journal (October 1958): 613.

[2] John R. Sweeney, "An Experimental Study Comparing the Cuisenaire Method with Traditional Methods in Grade 1 Mathematics," in Canadian Experience with The Cuisenaire Method, ed. Canadian Council for Research in Education (Ottawa: The Council, 1964), pp. 117-140.

Research by Hollis[2] compared the traditional and the Cuisenaire-Gattegno methods in grades one and two.

---

[1] E. N. Ellis, "The Vancouver Experiments," in Canadian Experience With The Cuisenaire Method, ed. Canadian Council for Research in Education (Ottawa: The Council, 1964), pp. 13-23.

[2] Loye Y. Hollis, "A Study to Compare the Effects of Teaching First and Second Grade Mathematics by the Cuisenaire-Gattegno Method with a Traditional Method," School Science and Mathematics 65 (November 1965): 683-687.

Forty control and forty experimental subjects had had either one or two year's instruction, depending on grade, in the respective methods and the students were matched by defining five I.Q. levels. There were no significant differences found in the first grade scores but the second grade Cuisenaire group performed significantly better at the .01 level on a standardized test. Hollis notes that both the first and second grade children using the Cuisenaire material acquired additional abilities. Here again the use of a standardized test for comparison purposes only permits mentioning the additional abilities of children which have been acquired, and which might be demonstrated on a different instrument.

Passy,[1] (1963) reported a study undertaken to compare the differences in achievement on tests of reasoning and computation. The study used 990 experimental subjects and an aggregate of 843 control subjects: 376 contemporaries and 467 from an old set of scores on the identical standardized test given in grade three. The test used was the Stanford Achievement Test, Elementary, complete battery. Passy cross-matched the subjects by reading level (Stanford) and by attendance in school. He rejected the null hypothesis when he found a significant difference ($P < .05$) in the scores favoring the non-Cuisenaire group. However, the validity of the Stanford Test for comparing these two groups is questionable. The outline of the Cuisenaire program, a modified one generated by the school district, not identical to the Cuisenaire-Gattegno method, follows: unit 1, factoring; unit 2, operation cross, (multiplying); unit 3, planetary arithmetic, (bases); unit 4, solution sets; unit 5, operation signs, (determining the proper signs to make an expression an equation); unit 6, fractions.[1] This outline is not at all congruent to a traditional outline and is especially incongruent with the Stanford test inasmuch as only two of the units are tested in the elementary battery: multiplication and fractions. One could reasonably expect to find significant differences in favor of a group who had studied the material.. Also, Williams' statement regarding the questionable value of testing structural methods with traditional tests is relevant to this research. Finally, a question arises regarding whether the program designers were fully conscious of the nature of the differences among their program and a traditional one. Passy notes that there may be a question regarding the usefulness of the Stanford test but he points out that there was no test for the Cuisenaire program as designed.[2] Glennon and Callahan,[3] (1968) who report on Passy's research, make a two-fold error. First, they do not make clear that Passy's comparisons were not between rods as enrichment, and traditional methods.[1] And, secondly, their willingness to accept Passy's results as valid means that they count the Stanford Battery as valid for assessing a structural program. Thus, while some of the research on the rods used standardized tests and found no significant differences, Passy's similar research using the same kind of tests has been used as evidence for questioning the usefulness of the rods as he did find significant differences favoring the non-Cuisenaire students. Yet, the result of this effort gets reported in a summary volume without any mention of the necessary questions pertaining to the validity of the measures for the methods being compared. The cause of the error might be due to a careless reading, or a lack of awareness of the problems Williams identified. In either case, it seems clear that there needs to be more care taken in summarizing reports.

---

[1] Robert Albert Passy, "How Do Cuisenaire Materials in a Modified Elementary Mathematics Program Affect the Mathematical Reasoning and Computational Skill of Third-grade Children?" (Ph.D. dissertation, New York University, 1963).

---

[1] Ibid., pp. 23-24.

[2] Ibid., p. 98.

[3] Vincent J. Glennon and Leroy G. Callahan, Elementary School Mathematics A Guide to Current Research (Washington, D. C.: Association for Supervision and Curriculum Development, NEA, 1968), pp. 21-22.

Another format for researching the usefulness of structural materials employs the teaching of one or two specific topics taken from the curriculum. These topics are then presented to students, a control group using the traditional approach, and the experimental group using Cuisenaire rods or an adaptation of the Cuisenaire-Gattegno method. The problem with this format is that the rods have the possibility of modeling all of the four operations (addition, subtraction, multiplication, division) and the interrelatedness of the operations. This, as Gattegno has pointed out,[1] is one of the primary reasons for using them as mathematical manipulatives. By using them for only one topic, much of their value is lost. Further, the amount of familiarity with the rods that is gained over time enhances the students' possibilities of seeing and understanding the various operations.

---

[1] Ibid., p. 22.

Lucow,[2] (1962) sought to determine the amount of growth that might be seen in a six week experiment in the teaching of multiplication and division to third grade children. Because of the fact that the Cuisenaire children had had experiences in these operations in first and second grade Lucow adjusted the means of the pre-test scores to attempt to answer the question,

> What would the growth mean have been had there been no difference in pre-test scores between the Cuisenaire and control groups, and is the difference between the adjusted growth means significant?[3]

As a result of the pre-test scores and the scores on the Otis-Lennon Mental Abilities test, Lucow divided his subjects into Alpha and Beta blocks such that Alpha block included 66 experimental and 53 control subjects while Beta block had 63 experimental and 72 control subjects. Lucow classified his data in the following ways: Bright, Average, Dull, Boys, Girls, City, Country. Overall, the results favor the Cuisenaire group but more importantly the results show that the operations of multiplication and division may be taught in a much shorter period of time than had been thought possible. As the Cuisenaire children had been introduced to the topics in earlier grades, it is possible that the momentum of their previous study aided them during this experiment. At the same time, it is also possible that the type of examination (criterion test developed by Lucow) favored the controls in that it was within the traditional style of quick response making rote recall of details advantageous.

---

[1] Gattegno, For The Teaching of Mathematics, 3:49. See chap. III, p. 67, above.

[2] William H. Lucow, "Cuisenaire Method Compared With Current Methods of Teaching Multiplication and Division in Grade 3," (Winnipeg, Manitoba: The Manitoba Teachers' Society, 1962).

[3] Ibid., p. 6.

Inasmuch as the Cuisenaire students had been introduced to the full range of rod uses, the problems of topical experiments are not so pronounced in the Lucow study as they are in a study done by Haynes,[1] (1963).

Haynes designed research to compare the use of the rods for teaching multiplication and a traditional method for the same topic. Neither the teachers nor the students had had prior experience with the rods. Haynes adapted the Cuisenaire-Gattegno curriculum and created a teacher training program. The students were given a three-day period for free-play with the rods prior to the beginning of a six-week unit on multiplication. A total of 106 children in five groups, two Cuisenaire, two control, one non-aware control, were given a criterion test as well as the spatial and numerical sub-tests of Thurstone's Primary Mental Abilities Test. Haynes' study found no statistically significant differences among the groups. However, the structure of the rods is such that multiplication is most clearly established as an operation through the study of repeated addition. Further, the products or sums of any trains (rods placed end to end) are established by measurement. Therefore, the use of the rods for multiplication only is not useful because that study requires the coincident study of addition and measurement. If this is not done, the structural properties of the rods will have no value to the learner.

---

[1] Jerry Oscar Haynes, "Cuisenaire Rods and the Teaching of Multiplication to Third-grade Children," (Ph.D. dissertation, Florida State University, 1963).

Another consideration arising from Haynes' study relates to the amount of teacher training necessary if manipulative materials are to be useful. When teachers are new to the material, there tends to be some hesitancy to allow the material to be itself. Without experience, a teacher does not realize that the rods are capable of representing only true facts of multiplication. And, prior to this realization, teachers will be as directive with the rods as they were when they were using purely traditional, pencil and paper techniques. The dialogue that the student might enter into with the rods is not possible under those conditions.

Three studies done in Great Britain provide further evidence regarding the importance of teacher training.

Biggs,[1] (1959) monitored the progress of 4,040 students in various math curricula, 899 of whom had instruction within a structural curriculum. Using analysis of variance, Biggs found no differences regarding average children using Cuisenaire or other similar structural materials. Further, he found no evidence for using the structural materials of the Cuisenaire type with dull children although there was some indication that they might be helpful for use with bright children. Biggs mentioned that it was difficult to know the true effectiveness of the material because it was new and teachers had little experience with it. Williams[2] compared two structural methods, Cuisenaire and Dienes, with non-structural methods. He found that pre-testing increased scores on all measures especially when the pre-testing occurred after a period of practice with the new materials or methods. This could be attributed to the role that tests play in setting objectives when a new method is introduced. Teacher and student experience can be seen as important when we read from Williams that:

> After a year's use of the Dienes method, first-year children's scores are slightly depressed by comparison with those of their controls. The corresponding first-year Cuisenaire children, however, obtain slightly higher scores than their controls on all measures excepting Attitudes, on which scores are significantly lower than those of the controls. But

---

[1] J. B. Biggs, Mathematics and the Conditions of Learning (Slough, England: National Foundation for Educational Research in England and Wales, 1967).

[2] John D. Williams, "The Evaluation of Three Maths Courses," Journal of Structural Learning 3 (1972): 41-79.

> after a further year, those Dienes and Cuisenaire children tested at the end of their first year, improve on all three tests to a level significantly above that of their respective controls. The Cuisenaire children, moreover, improve significantly to a new level on the attitudes scale.[1]

These results demonstrate clearly the need for practice and experience. Another factor in determining the effectiveness of various curricula is teacher quality. While quality is indeed affected by experience it is also a function of interest and ability. Brownell,[2] (1966) in Scotland, found significant differences favoring children using the Cuisenaire method over a conventional one, in overall mathematics achievement. In similar research, conducted in England, the results were just the opposite. As the programs were similar, Brownell noted that the organization of a program on paper was no indication of what was taking place in the classroom inasmuch as the quality of teaching might "... range all the way from excellent to poor."[3] His explanation for the differences in achievement was that:

> ... the Cuisenaire program was better taught in the Scottish schools than in the English schools, and the conventional program was better taught in the English schools than in the Scottish schools.[4]

---

[1] Ibid., p. 78.

[2] William A. Brownell, Arithmetical Abstractions; The Movement Toward Conceptual Maturity under Differing Systems of Instruction (Berkeley and Los Angeles: The University of California Press, 1967).

[3] Ibid., p. 194.

[4] Ibid.

Whether these differences are due to interest or ability is not mentioned in the report. Biggs, Williams, and Brownell have reported research which has pointed to the teacher as needing experience, interest, and ability. Further, they have shown that an indication of a set of goals is possibly helpful to teachers engaged with a new material. One reason for this might be that with the structural materials, many directions for studying mathematics become apparent and available to the teacher. When a pre-test has been given, the course of study takes a specific direction related to the content of the pre-test. However, it should be noted that this direction is not necessarily the right direction. It is only one of many directions that could have been taken. In effect, what the pre-test does is to aid the choice of direction. And, when the teachers are aware of the direction and are interested, experienced, and able, students show improvement on test scores.

One researcher, Aurich[1] (1961) reported on an experiment to compare the Cuisenaire method with a traditional method in the first grade. Using 45 control and 45 experimental students from parochial schools Aurich matched the students by age, sex and scores on the S.R.A. Primary Mental Abilities test. The instrument used was the S.R.A. Achievement series, arithmetic 1 and 2, form A. The results of the comparison favored the Cuisenaire boys at the .01 level in concepts, computation, and reasoning. The Cuisenaire girls showed significant differences at the .01 level on concepts, and computation and at the .05 level on reasoning. These results are true for both a June and a September administration of the tests. These results are not typical in that a traditional measure was used to compare structural and non-structural curricula and the results favor the structural approach. What the results probably represent is the type of experimental stimulation cited by Williams. Evidence of such stimulation can be found in the decision by Aurich to have teacher-made tests constructed to reflect more adequately the content of the respective courses of study. This was done because

---

[1] Sister Mary Richard Aurich, "A Comparative Study to Determine the Effectiveness of the Cuisenaire Method of Arithmetic Instruction with Children at the First Grade Level," (M.A. dissertation, Catholic University of America, 1963).

> ... the standardized test did not adequately measure the mathematical processes explored by the boys and girls in the E group.[1]

A total of sixteen tests were constructed by the teachers, eight reflecting the control content, and eight reflecting the experimental content. On every test, whether control or experimental, the experimental groups of boys and girls performed significantly better than their counterparts. However, it is likely that the experimental teachers were well informed of the control content while this is probably not true of the control teachers for the experimental content. Another possible explanation for the results is that the teachers were not of equal ability. While the students were carefully matched, there is no information regarding how the teachers came to be part of the experimental program. If teachers volunteered, then interest was obviously high. In summary, the results of the Aurich study must be viewed in the light of possible differences among teachers in experience, interest and ability.

---

[1] Ibid., p. 45.

Sweeney[1] (1964), in an attempt to compare the rods with other, non-structural manipulatives, adapted the Cuisenaire-Gattegno course of study for use by all first grade teachers in the Kitchener Separate Schools. This was done in order to be able to say whether the rods themselves were significant in the teaching of higher level concepts (e.g. multiplication and division) or whether a course outline which included studies of higher level concepts would produce similar results. In order to avoid the problems of teacher interest, ability, and experience Sweeney and his colleagues rated all of the first-grade teachers and included their years of experience and preference for teaching method in matching a set of Cuisenaire teachers (using rods) to non-Cuisenaire teachers (using a non-structural manipulative). None of the teachers had had any experience either in teaching the curriculum content or in the use of the rods and the other manipulatives. Teacher training was provided for both groups throughout the year. Sweeney and his colleagues developed a set of reliable and valid tests for use with the entire first grade population, 400 pupils in twelve classes. (Note: two classes were eliminated due to changes in teachers and their matched groups also had to be eliminated from the results). No significant differences were found among the mean scores of the various structural and non-structural groups. On a set of fourteen tests administered from February to June more than half of the pupils tested scored over sixty percent correct on the tests and their scores were most often around eighty percent correct. Test items included addition, subtraction, multiplication, division, use of unknowns, inequalities, and word problems. Sweeney attributes the good results more to teacher involvement than to any specific method.[1] However, he does note that the specialized nature of the Cuisenaire material makes experience very important and it is possible that after some years experience with the material Cuisenaire teachers would have produced much superior results.[2] The value of this research lies in the fact that it points to the possibility that many more topics can be taught to young children with the use of a manipulative aid. It also shows the kind of results that can be achieved with thorough and ongoing teacher-consultant interaction. The most important point to be made in regard to this study is that in order for a school or school district to design a course of study which includes all of the topics mentioned above it must have respect for the learning ability of the young student.

---

[1] Sweeney, "An Experimental Study Comparing the Cuisenaire Method with Traditional Methods in Grade 1 Mathematics," pp. 116-140.

---

[1] Ibid., p. 128.

[2] Ibid., p. 129.

Another study, Keislar[1] (1961), approached children with the same kind of respect as that evidenced by the Kitchener curriculum. Keislar evaluated the ability of first grade children to learn algebraic structures, inequalities and unknowns, with the use of teaching machines and rods. The study used 136 subjects from Los Angeles schools. An additional group of children not involved in the experiment were tested in order to know what first-grade students might learn about algebraic structures in the absence of instruction. The children who were taught received less than fourteen minutes per day of instruction yet, by the end of the program they were able to produce equations and inequalities by manipulating plastic symbols. Keislar quoted one of his colleagues regarding the relative values of the manipulative material and the teaching machine:

> The intrinsic reinforcement gained through visual and tactile senses proved to be far superior to extrinsic reinforcement supplied by machine with green light and verbal confirmations. This was clearly indicated by the expressions on the pupils' faces at the time when they discovered the solutions with their sticks and the anti-climatic [sic] reaction to the reinforcements of the machine.[2]

---

[1] Evan R. Keislar, "Abilities of First Grade Pupils to Learn Mathematics in Terms of Algebraic Structure by Teaching Machines," (Los Angeles: The Cooperative Research Program, U.S.O.E. Project SAE 8898 No. 1090, University of California, Los Angeles, 1961).

[2] Ibid., p. 64.

There were three types of tests given at the end of the program: 1) creative; 2) solving problems, and; 3) applications. On the creative test, one item asked children to make as many true stories as possible using the symbols: 2, 4, 6, +, -, <, >, =. There were five problems of this type, some using unknowns (X) and some using only inequalities. The mean number of correct responses was 20.6. A group of 58 fourth graders who had not received instruction had a mean of 21.0; a group of 95 first graders who had not received instruction had a mean of 9.9.

On the problem solving test children were asked to find the correct value for X from a set of three possible answers for such expressions as, $3+5=X$, $X+X>7$, and $2\cdot(4+1)=X$. There was a total of 18 items of this type and the means for the three groups were: 15.2 for first grade experimental; 12.2 for first grade control; 19.5 for fourth grade control.

The application test used items which provided the children with information about a picture and then asked for deductive conclusions. There were 12 problems and the means were: first grade experimental 7.5; first grade control 6.8; fourth grade control 8.3.

While there are limitations in this study, (no control for I.Q., problems of individual interview and individual tests) it does demonstrate what might be possible for first grade children to experience and express using a set of manipulatives. The teaching machine provided consistency of response but, as noted above, probably had little effect on enabling the children to learn.

Summary

One cannot draw any definite conclusions from the above mentioned study. However, one can note the possibilities that this study points to regarding mathematical learning and the usefulness of manipulatives. In fact, all of the reports included in this survey have sufficient limitations so as to make them less than definitive. In sum they all present evidence of the possibility that mathematical learning may be enhanced by the inclusion of manipulatives in the teaching of the subject, and by an increase in respect for children's ability to learn evidenced by an increase in the amount and scope of topics presented to them. Reports have been presented where the use of a standardized test has not produced much information about the effectiveness of the rods. Reports have been presented where teacher preparedness, and/or teacher student preparedness has affected the outcome of the evaluation. And, reports have been presented where interest probably effected a great increase in teacher and student work levels. Future researchers will do well to heed Williams' cautions and create a design which alleviates the problems. Also, as noted earlier, there needs to be research into the use of rods as tools for enriching the mathematics curriculum as it is currently defined.

Chapter IV

A CONCRETE MANIPULATIVE FOR TEACHING NEGATIVE NUMBERS

Introduction

Chapter three reviewed the literature with which one can assess the usefulness of the Cuisenaire rods. While it cannot be concluded from the data available that the rods are absolutely beneficial, one can infer that the rods have the possibility of providing children with experiences not available when manipulatives are not used. It is true that another manipulative may have performed equally well or better but Cuisenaire rods have been chosen for their popularity, their usefulness, and because of the possibility of extending the rods into the negative numbers.

The survey of present practice in the teaching of negative numbers (chapter two) concluded with the statement that a concrete manipulative should be so structured that it retained its negative characteristics, or its relationship to negative numbers across all activities associated with play. None of the materials surveyed was able to do that. One of the reasons why this is true is that none of the materials for negative numbers has been conceived from the point of view of oppositeness. That is, where a material was said to be negative, this was only by definition, not by nature or relationship to a positive material. To overcome this problem the Cuisenaire rods, a valid and useful tool for representing positive numbers, will be extended. With the rods representing the positive numbers, it is only necessary to conceive of a shape which will act as the opposite of the rods, and will therefore carry negative meaning.

This material takes a simple shape. It is an upside down "U" with interior dimensions of 7/16 of an inch by 7/16 of an inch by 7/16 of an inch. With this interior dimension the rods will slip in and out of the structure easily. Also, the material will slip on and off the rods without friction. (Note: The exterior dimension will be a function of the thickness of the material used.) The rods have length, one cm to ten cm. The negative materials have identical lengths. A set of negative rods will contain materials in lengths corresponding to the lengths of the positive rods with a minimum of four of each length.

An important attribute of the Cuisenaire rods is that they have color. For the negative material, color should probably be used when it is to be used with students who have experience with the Cuisenaire rods. When the negative rods are being introduced along with the introduction of the Cuisenaire rods the negative material should be marked in some way so that each length is easily distinguished. Where the possibility for mass production exists, the negative material should be made of clear plastic, lucite or butyrate, with notches across the top of each rod at centimeter intervals. While these notches impinge on the nature of the rods (counting is not recommended) they will make both materials more useful for students who have not had experience with them. Figure 5 shows the Cuisenaire rods and the negative material side by side.

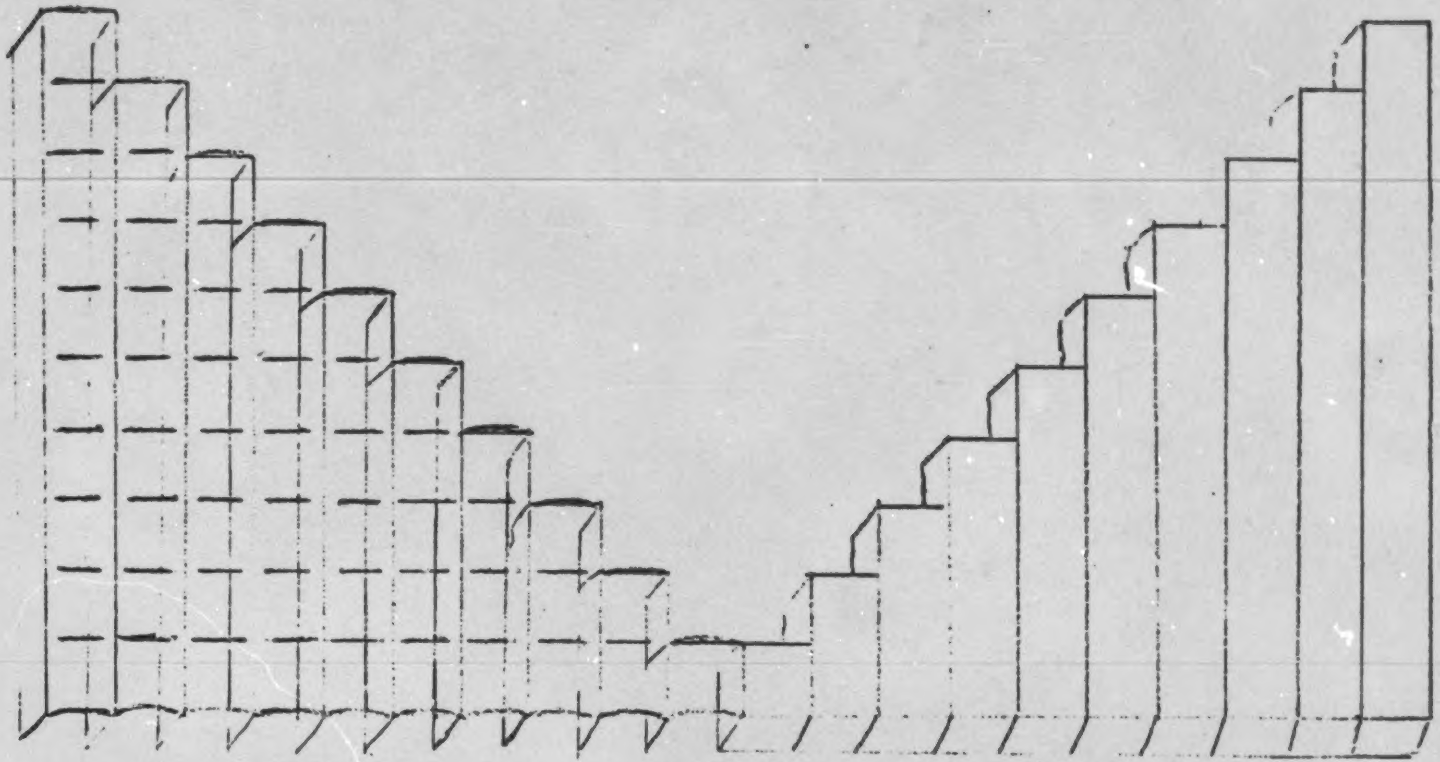


Fig. 5. Negative rods and Cuisenaire rods.

With some idea of how the material looks, it is possible to begin to examine the operations with it.

Operations

As stated earlier, the material has been designed to be the opposite material to the Cuisenaire rods. That is, if a rod of two centimeters is considered to be positive two ($^{+}2$), then a material of two centimeters which, in some way acts as the opposite of that material by covering or consuming the length may be considered negative two ($^{-}2$). With an opposite length to every centimeter rod, it is possible to have any negative rod covering a part of, or more than, the length of any positive rod. When this is the case, one may note the uncovered or unfilled length (i.e. the difference) and name the number. Where there is no difference between the two lengths (i.e. totally filled or covered), one may note this assemblage as zero.

The act of addition with centimeter rods is the making of a train; such is also the case with the negative material. To add two negative lengths, one simply places these two lengths end to end and notes the new length. To add a positive rod to a negative length, one again makes a train. However, in this case the cars are of different material, and one must push one car inside the other as far as it will go. Then note the length of the difference and name the result. Of course, order in this operation does not matter. Likewise when there are three addends, one makes the train, performs the pushing operation as needed, and notes the resulting length. Figure 6 depicts the addition of two negative quantities with the negative rods. Figure 7 depicts the addition of a positive and a negative quantity using both the Cuisenaire rods and the negative rods.

$^{-}4 + {}^{-}3 = {}^{-}7$

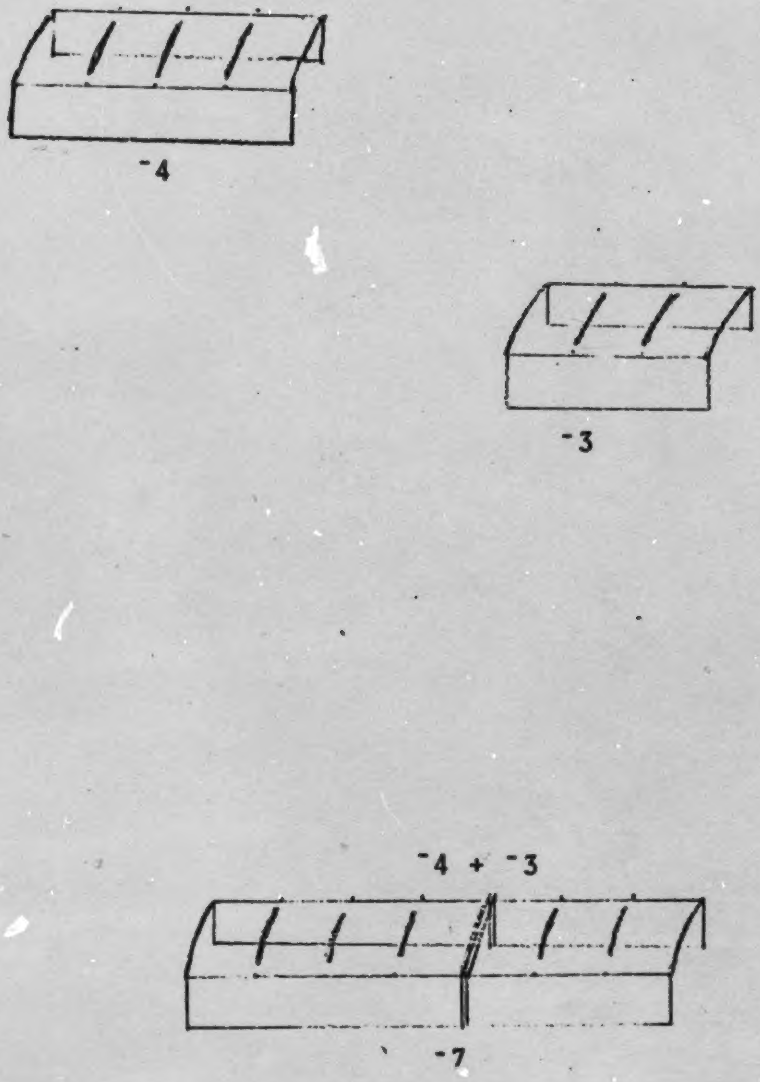


Fig. 6. The addition of two negative quantities.

$^{+}6 + ^{-}2 = ^{+}4$

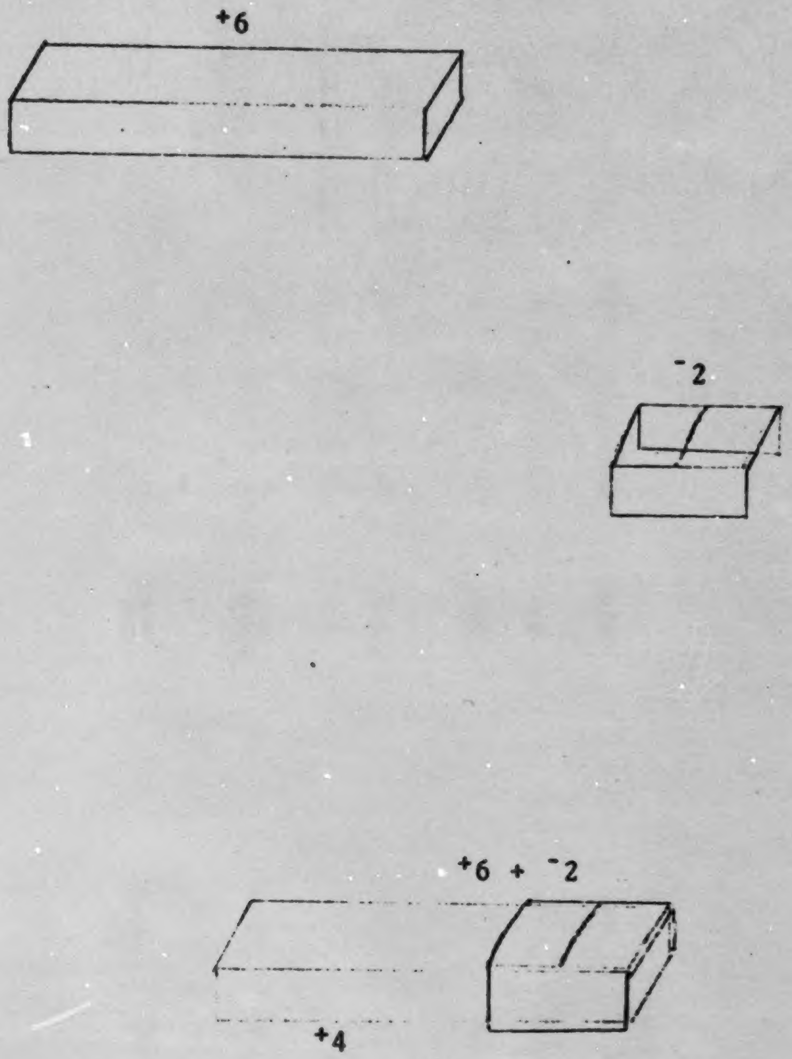
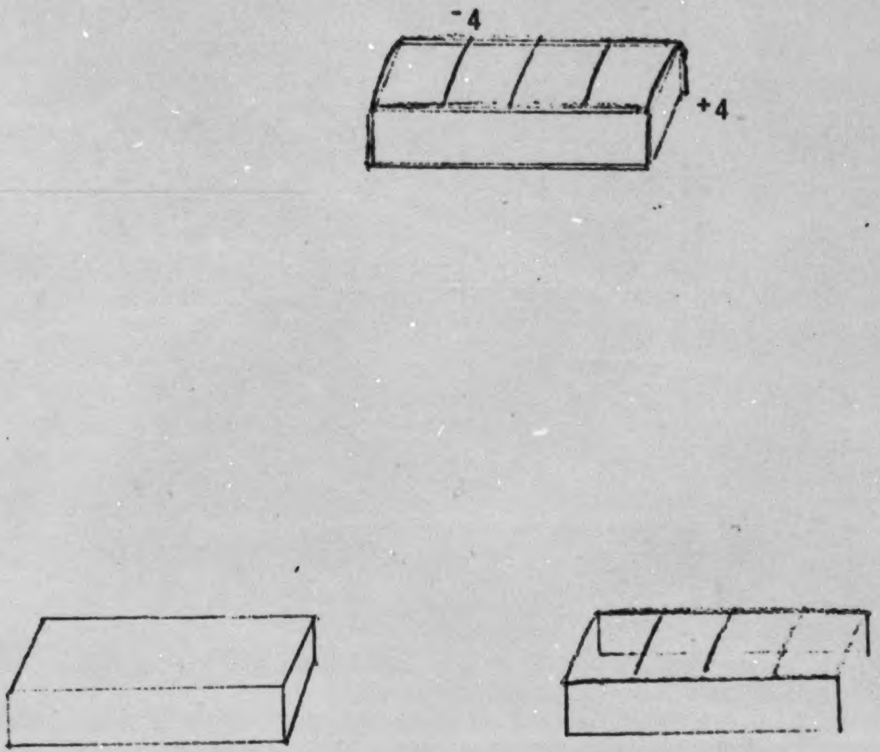


Fig. 7. The addition of a positive and a negative quantity.

For subtraction the language of "take away" when working with the negative material is helpful. If one assembles a zero using the five centimeter lengths, one can take away either the positive rod or the negative length from the assemblage and note the result. It is obvious that what is to be taken away from zero dictates the choice of the zero assemblage. When the minuend is not zero, one is still able to use the "take away" language by adapting the minuend to the possibility of "taking away" the subtrahend. This is possible because of the characteristics of zero. When zero is added to any number, it has no effect so zero can be added to any minuend. Again, the choice of the zero assemblage is dictated by the subtrahend. For example, if there is a positive three rod from which one needs to subtract a negative eight, he can add a zero assemblage, eight centimeters in length. It is now possible to "take away" the negative eight leaving a positive length of eleven centimeters. The activity is the same for subtracting from any kind of number; the minuend is adapted by the addition of a zero the type of which is dictated by the subtrahend. The situation where the minuend and subtrahend are of like signs uses the same adaptation but is followed by the pushing operation referred to in the addition section. Figures 8, 9, and 10 depict three subtraction activities.

$0 = {}^{+}4 + {}^{-}4$



$0 - {}^{-}4 = {}^{+}4$

$0 - {}^{+}4 = {}^{-}4$

Fig. 8. Subtraction from zero.

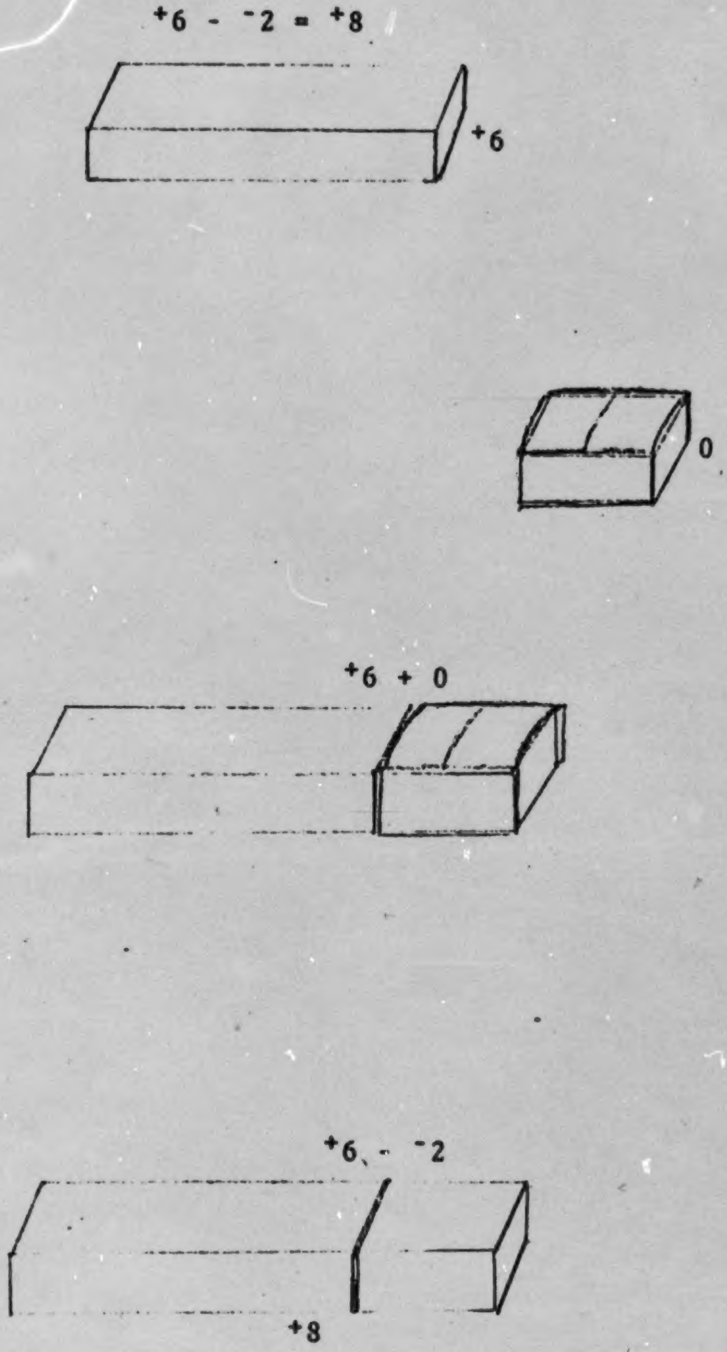

+6 - -2 = +8
+6
0
+6 + 0
+6 - -2
+8

Fig. 9. Subtraction.

$^{+}3 - {}^{+}5 = {}^{-}2$

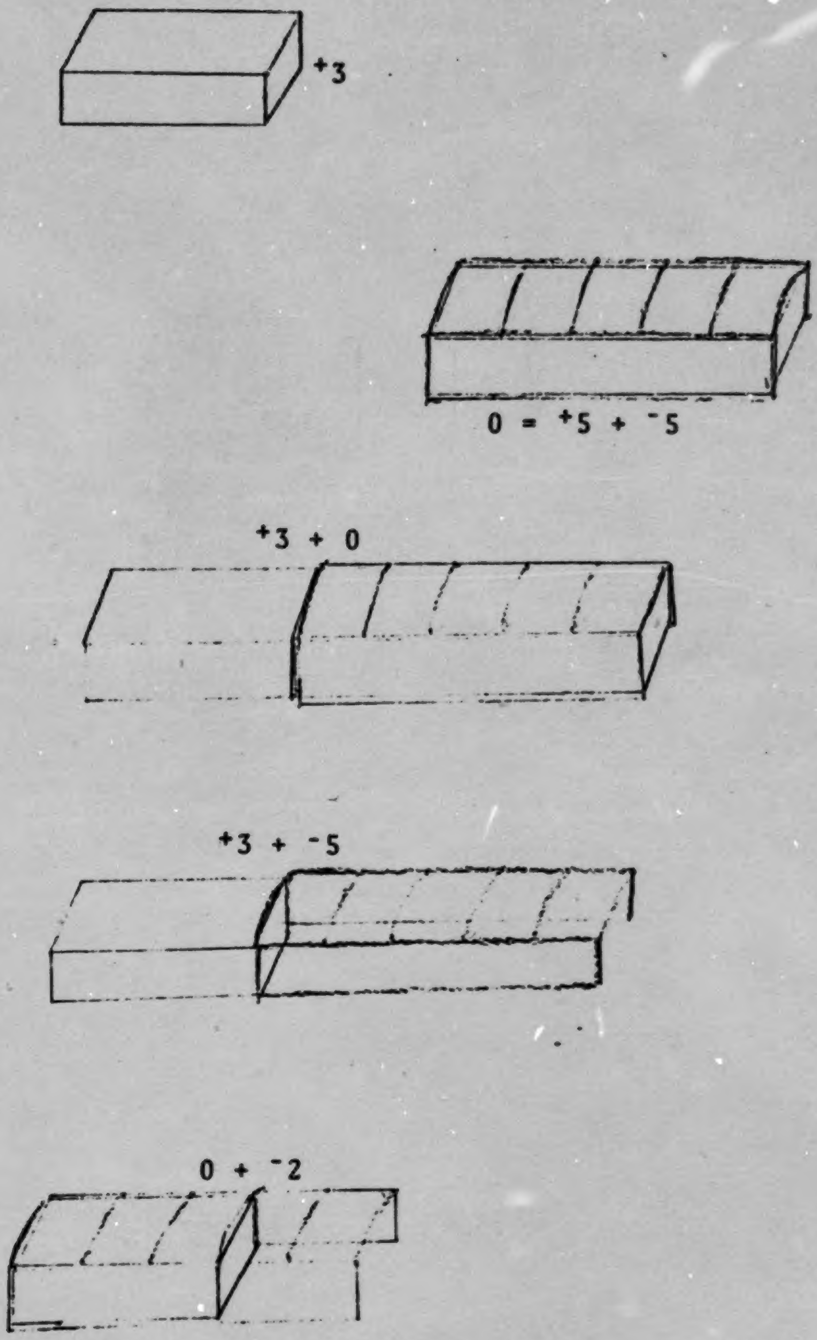

+3
0 = +5 + -5
+3 + 0
+3 + -5
0 + -2

Fig. 10. Subtraction.

The manipulations which have been described above, show that it is quite easy to reconstruct the actions mentally thereby entering into virtual activities. The abstraction from the physical to the virtual is from an action. The action performed is directed by the sign or signs within an expression. It might be said that the signs refer to actions -- manipulations; one reads the signs as directions for actions. When the situation is not suitable for the directed action, the situation is adapted in order to make possible the indicated action. The signs inform the reader of the way to simplify the expression. If one can find ways of reading the signs which are both legitimate and informative, then it seems right to do so.

In the universe of subtraction there are at least three ways to read the sign (-); it can be read "subtract," "minus," or "take away." Clearly the most informative of these decodings is "take away" especially when working with manipulatives.

In the universe of multiplication there is also found a variety of ways to read expressions; 6 x 4 may be read "six multiplied by four," "six times four," or "six fours." While the product is not affected by the readings, the action is affected. If one chooses to read "six multiplied by four," the actions must be to pick up a six, again, again, and again. Conversely, reading "six times four," the action might be to pick up a four, six times. If one reads the simpler, "six fours" then the expression is informative and the action is clearly implied within it. With this as a possible way of reading, one can read the expression, 6 x $^{-}4$ as, "six, negative fours." The action that is required is the ordering of six negative lengths of four centimeters. The result of the action is a negative length of 24 centimeters.

The expression, $^{-}6$ x $^{-}4$, might be read, "negative six, negative fours." However, this does not speak to us of an action. Is there a way to read any of the signs within that expression that will provide direction regarding the necessary action? There is. By returning to the simple language of subtraction when reading one of the (-) signs, one may hear the directions for the action. By re-reading the expression, this time substituting "take away" for "negative" at the first (-) sign, the expression says: "take away six, negative fours." The action to be performed is clear, but what is it that is being taken away from? An assemblage of zeros is always available for the indicated action. Again, these zeros are dictated by the expression. If there are available sets of zeros of the form ($^{-}4$ + $^{+}4$), it is possible to "take away" from that set six negative fours resulting in a positive length of twenty-four! If commutativity is not employed, the same operation works for the expression, $^{-}6$ x 4. Here one reads, "take away six, fours." From the zero assemblage he takes away six fours giving a negative length of twenty-four.

While it is simple enough to conclude what sign a quotient should have from the knowledge of multiplication, one can also manipulate the negative material in this operation. It is known that zero is undefined for division. That is, some number divided by zero has no quotient. This can be demonstrated by making a positive length and choosing any set of zeros to measure it. It can be seen that it could be any zero and when the situation arises where something could be anything, it is said to be undefined.

The sign (÷) is not very informative when read "divided by" because it is not clear what the action should be. For example, in 12 ÷ 4, should one make four groups or should one make groups with four in each? If, however, the sign is read informatively as, "has how many," one can know what to do. "Twelve has how many fours" means, "make the twelve into groups of four and count how many groups are created." With rods, it is clear that it is necessary to measure the 12 cm train with 4 cm trains.

For those division situations where the sign of the divisor and dividend are the same, the problem is solved by finding "how many it has" by measuring. When the signs are different, it is possible to make a zero train of a type related to the divisor. The answer reflects the necessary operation to change the zero train to match the dividend train. For example, for the expression, 12 ÷ ⁻4, make a zero train (4 cm) and see that in order to match the 12 it is necessary to take away three of them, encoded ⁻3. For the expression, ⁻12 ÷ 4, again make a zero train and see that it is necessary to again take away three of them, encoded ⁻3.

Relationship to Theory

With some idea of how the material works in the four fundamental operations (addition, subtraction, multiplication, division) it is time to classify it according to the definitions of chapter two, and to comment upon its relationship to the theories that were outlined in chapters one and three.

The negative rods method uses the physical mode and employs both counting and measuring to evaluate a sign and a product. Counting and measuring are used at different times; counting takes place when there exists a set of discrete zeros and an action (a counting one) is to be taken on them; measuring takes place when a set of negative rods is ordered such that their value may be known by determining the length of a train. The combination of possibilities makes both addition and subtraction, and multiplication and division available with the same rules and the same material. Some of the other methods which were examined in chapter two seemed to be intended for verification of the product of two negatives. While the manipulation of the negative rods can also serve this purpose, it is clear from the presentation of the operations that it is well suited to establishing the rules for negative number multiplication. The problem of the "inner reasonableness" of $(^{-}1)\cdot(^{-}1)$ is alleviated when the signs speak to the learner of action, and when that action makes sense to him. With the negative rods, the taking away of one negative one only requires that there be a pre-existing set of zeros so structured that the action called for is possible. It was noted in chapter two that the problem of "inner reasonableness" was probably an individual one, a matter of response by particular students. The negative material presented in this chapter will make it possible for a large number of students to respond to their action in a way that allows them to come to "inner reasonableness" regarding the product. (Note: In chapter five, certain guidelines for student and teacher activities will be presented.)

The response is possible because the material simplifies the information, makes the generating of new propositions well within the province of the learner, and is totally manipulable. These three criteria will be recognized as those defined by Bruner[1] for weighing the merit of a structure. It should be kept in mind that the structure referred to by Bruner is not necessarily a physical one and that in this case it includes both the material and the attendant language.

The information to be known regarding negative numbers includes the facts that they are opposite to positive numbers, and that they admit to four fundamental operations. This sample of the information to be known demonstrates clearly that the negative material does serve to simplify the information. It does so by being in relation and, to some extent, in essence negative and therefore permits actions which when encoded can be related to the four operations.

---

[1] Bruner, Toward a Theory of Instruction, p. 41. See chap. I, p. 16, above.

Inasmuch as the above is possible, it can be predicted with some certainty that new propositions can be generated. All of the work outlined in the earlier section on operations used whole numbers. However, a student might decide to try to compute with the rationals or a combination of fractions with whole numbers. For example: $^{-}1/2$ x $^{-}4$. A student would decode this expression for the purposes of action thus: "take away one-half of negative four." He would then adapt a zero so that he could perform the indicated action which would be as follows. It is clear that the zero shall be four centimeters long and that the negative portion of it shall be in two equal parts. In symbols: $0 = 4 + {}^{-}2 + {}^{-}2$. After taking away one-half of the negative four it can be seen that positive two is the result.

The above example also demonstrates the manipulability of the material and the language. With experience it becomes easy to replace one assembly with an equivalent one, and the decoding for action is natural. That is, where guided activities have been preceded by play, many actions will have taken place. Some of them will be capable of being encoded and therefore decoded so that the symbols, whether written or verbal, speak of actions as well as relations.

This brings us to the question of play and action. Piaget[1] and Dienes[1] both spoke of the need for it and the survey of present practice showed that it would be difficult to provide. The difficulty resided in the fact that the materials did not admit to translations in space if they were to remain negative materials. This problem has been overcome in this design so that all of the benefits of play and action that have been related to positive numbers are now available for negative numbers when students are working with the Cuisenaire rods and the negative material. This is so because the material has been so structured that it relates to the structures of the mathematics of negative numbers. Further, this structuring permits the very same dialogue that Gattegno[2] speaks of to take place when the student comes into contact with the negative rods. The dialogue with this material consists of the student's exploring all that the material admits to. In other words, by using the material the student approaches the limits of its use. It might be said that the physical nature of the material speaks to the student. It speaks to the student regarding limits, possibilities, and relationships. When the dialogue obtains the very same "correct algebraic dynamism"[3] which Gattegno speaks of will be taking place. That is, the four operations available when this material is used in conjunction with their positive counterpart, Cuisenaire rods, will be experienced.

---

[1] Piaget, Six Psychological Studies, trans. Anita Tenzer, ed. David Elkind, pp. 127-128. See chap. I, p. 4, above.

---

[1] Dienes, "Some Basic Processes Involved in Mathematics Learning," pp. 21-34. See chap. I, p. 11, above.

[2] Gattegno, For The Teaching of Mathematics, 3:15. See chap. III, p. 67, above.

[3] Ibid., 3:15. See chap. III, p. 67, above.

This play and concurrent action are capable of preparing the student for the logic of the operations which are to be presented. The material is simple enough so that it can be worked with easily. This means that the student can begin to concentrate on his actions rather than on the material itself thereby exiting the realm of the physical experience and entering the realm of the logico-mathematical experience.[1] He can proceed to the activities which have been described as virtual.[2] In other words, he can omit an actual action when he is certain of what the results of that action will be. He can create a class by performing similar manipulations over differing lengths.[3] That is, by finding each of the negative rods which totally covers a related positive rod the student will have created classes consisting of opposites. These activities comprise the three views of abstraction presented earlier. If all three views are part of the act of abstraction, all three are available. In other words, abstraction, in any of the above definitions, is possible with the rods and the negative material.

---

[1] Piaget, Psychology and Epistemology, trans. Arnold Rosin, p. 71. See chap. I, p. 5, above.

[2] Gattegno, For The Teaching of Mathematics, 3:16. See chap. III, p. 72, above.

[3] Dienes, "Some Basic Processes Involved in Mathematics Learning," p. 22. See chap. I, p. 12, above.

This concludes the presentation of the new product, the operations available with it and its opposite, Cuisenaire rods, and its relationship to certain theoretical considerations. What is necessary now is some indication of how the product might be used with students. These guidelines will be presented in chapter five.

Chapter V

GUIDELINES FOR TEACHERS

Introduction

It has been shown that the concrete manipulative for negative numbers is well related to the implications of theory. This chapter will present a set of guidelines for practice which are the result of the field tests which have been conducted throughout the development of the negative rods.

Servais has stated:

> The greatest pedagogical value of concrete material is in allowing the pupil to gather his mental experience at his own pace without the authority of the teacher.[1]

This is an important point to keep in mind while working with the negative material. The teacher is no longer the arbiter of right and wrong because the positive and negative rods provide that feed-back immediately. A student does not have to be told that the positive four rod is covered by the negative four rod, he can see it. If teachers are to begin to make concrete materials as useful as they can be, they must scrutinize their role and determine what can only be provided by the teacher, and what is available to the student individually in operating with the material. Among the things which teachers must provide are conventions and language. For example, it is necessary to push positive rods and negative rods together when they are added. This might be determined by the student, but it is a convention which the material needs for proper use. Likewise, naming the rods positive and negative is a language use which is arbitrary but which has to be done. Once these conventions and others like them are established the teacher can take a position in the background. The child can determine through his manipulations the "answers." The teacher remains ready with new challenges as he sees the child exhausting his explorations with what he already knows.

---

[1] Servais, "The Significance of Concrete Materials in the Teaching of Mathematics," p. 207.

Bruner, in speaking about curriculum generally, notes that it must reflect both the nature of the knowledge being worked on, and the nature of the learner and his knowledge-getting process.[1] This is also true for teaching. The negative material along with the conventions for its use reflects the nature of negative numbers. The learner and his processes can be known only through observation. This defines a two-fold role for the teacher. He is in possession of the conventions, and he is the observer who can respond to the nature of the learner and his activities. The teacher focusses his attention on the learner rather than on the subject matter. It might be said, with Gattegno, that the teacher focusses on the student in order to guide the student in his work on the subject matter.[2]

---

[1] Bruner, Toward a Theory of Instruction, p. 72.

[2] Caleb Gattegno, What We Owe Children (New York: Outerbridge and Dienstfrey, 1970), ch. I.

This type of teaching results in the student's doing mathematics rather than simply learning mathematics rules. The difference between simply learning, and doing is an important one and has been stressed by the Mathematical Association of America's Committee on the Undergraduate Program in Mathematics.[1] They state:

> Communication is of the essence in mathematics, and prospective teachers must pay special attention to all of the ways in which mathematics is most effectively communicated.[2]

One of the ways in which mathematics is communicated is by describing an assembly of mathematical materials. This is verbal communication and is such that it will require the communicator, in this case the student, to employ vocabulary which relates to the mathematics of the situation being expressed. That is, given some configuration of rods and negative rods, it is incumbent upon the learner to find the language patterns which will allow another person to duplicate the configuration and, in the absence of that other person, the communicator must begin to attach signs to his language so that it can be written, and eventually read. Obviously the teacher plays an important role here. The teacher helps to make the one to one correspondence between the students' expressions and the signs which carry the same meaning. The first activity in this regard must be expression.

---

[1] Committee on the Undergraduate Program in Mathematics, Recommendations on Course Content for the Training of Teachers of Mathematics (Berkeley, California: Mathematical Association of America, 1971), p. 20.

[2] Ibid.

Too often students are hurried into the use of signs. They have not achieved any fluency in expression, yet they are required to decode the expression of another. Mathematics education is often guilty of trying to teach a person to read who has not yet learned to speak.

The field tests which were undertaken to ensure the useability of the product were conducted with the above principles in mind. Consequently the guidelines which follow are not to be taken as anything more than a set of possibilities for the interaction of students, teachers, and materials. The games and activities which have been mentioned are written as if the teacher were talking with the students. This talking may be done with activity cards or any other medium which is capable of directing or focussing a student's attention.

Throughout the section there appear many references to free-play. Generally these sessions follow the introduction of new possibilities and are intended to allow the student the necessary time for his exploration at his pace. The questions which are recorded represent a possible response to student activities. Different students will work in various directions so that the questions presented here will not always be relevant to student responses. The last point to make is that within the first section of these guidelines there is no provision for writing or reading signs. Every direction, question, and response is verbal in order to allow the students to express what they find without the added problem of writing it. At the end of a set of verbal experiences one might logically begin to attach the common signs to the expressions in much the same order as the original introduction. However, one may also choose to introduce the signs concurrently with the initial activities. This is a matter of choice which should be guided by the age, experience, and ability of the learners. In fact, the guide is offered as something which may be wholly chosen or wholly disregarded. It should be read as a summary of a set of experiences which were directed by the principles outlined earlier. And, inasmuch as it is based on past experience, it might never be relevant again.

Free-play

The initial activity is free-play. This is student guided exploration which cannot be omitted. Both the positive rods and the negative rods should be available for building, patterning or whatever else comes to the mind of the student. In cases where there is no prior experience with Cuisenaire rods as much as one-half of the first few class times could be devoted to student exploration. Throughout this guide free-play will be mentioned in order to allow the students to experiment with the new possibilities introduced. These sessions are equally important and do not have to reflect the new possibilities although usually they will.

Games

Certain games may be developed and played with the students to increase their familiarity with the materials. For example, ask the students to take any three rods, Cuisenaire or negative, and hold them in their hands behind their back. Then ask them to show one of them, identifying it by touch, then looking to see if it is the one asked for. The number of rods in the hand may be increased and the content of them should be changed often. Students can play this game in groups without the teacher. Another game for familiarity uses a paper bag into which is placed a selection of rods which are then drawn out by using touch only. A third game begins with a set of positive rods on a table or desk at the student's right hand, and a similar set of negative rods at his left hand. His job is to pick up the rod in the left hand which is similar to the one he chooses with his right hand. All of these games increase the student's familiarity with the rods by requiring him to imagine them.

The guide which follows is presented in nine sessions. The organization of the sessions is arbitrary and the length of each of them will vary according to the experience of the students. The sessions should probably not exceed twenty minutes. However, if the students' interest is high the sessions could easily be extended.

Session	Topic	Page
1	Naming the rods - games	124
2	Addition	125
3	Zero - subtraction	127
4	Multiplication	128
5	Division	130
6	Symbols - writing names addition	132
7	Subtraction	134
8	Multiplication	135
9	Division	137

Dialogue

Session 1

Free-play.

Teacher: "If we call the white rod positive one, what shall we call the rest of the wooden rods?"

When the students have named all of the Cuisenaire rods, ask:

"What can you cover the white rod with so that it is totally covered but doesn't have any extra sticking out?"

Respond: "Because the one you found covers the white, we'll call it the opposite of white, and if we also call white positive one, what shall we call its opposite?"

Some students are likely to know and be able to use the term negative. If not, introduce the term and have the students begin to find all of the opposite rods to the positive rods, and then the opposite rods to the negatives.

Game: "Put three negative rods behind your back. (e.g. $^{-}6$, $^{-}8$, $^{-}10$). Who can hold up the opposite of positive eight? etc."

Repeat this game with different negative rods then switch to positive rods behind the back, asking for the opposite of negative numbers. Free-play should follow this introduction of vocabulary in order to give the students an opportunity to think about the rods with the new terms.

Teacher: "Who can find the rod that will cover all of positive six without any left over? Cover it. What shall we call this assembly?"

Give the students time to discuss the possible names they think are valid. If none of them is able to express and defend zero as a possibility establish it as the name.

"What will you give to positive eight to make zero, etc.?"

"What will you give to negative five to make zero, etc.?"

"What happens when we add opposites? Will it always be true?"

Session 2

Free-play.

Teacher: "What happens if we make a train, that is, add, two negatives?"

"Make some two rod trains, state what rods you used and what you call the new length." (Note: Where needed the same activity should be done with the Cuisenaire rods.)

"Choose any positive rod and add to it any negative rod. What happens now? What is your result?"

Some establishing of the "pushing" operation might be necessary here. Some students will immediately slip a rod inside the other and say that the new length is whatever has not been covered, or filled. This is another time for free-play. Several new possibilities have been established and the students should have the opportunity to explore them.

Teacher: "Take a positive three rod, what can you add to it so that your result is negative one?"

"What will you add to negative five to get a result of positive one?"

"Do you need to actually do it?"

"Pretend you have a positive seven rod, what would you add to it to get the result negative one?"

At some time following these experiences it might be useful to establish the relationship between rod size and numerical size. With the positive material, as the rod gets longer, its numerical value gets bigger. Inasmuch as the negative material is opposite to the positive rods, it seems reasonable that as the negative material increases in size its numerical value decreases. Another way to establish this rule is by pointing out the amount that each negative rod needs to make zero. The more it needs, the smaller it is; therefore negative ten is smaller than negative eight. Another way to establish the rule could be to choose a constant, positive four for example, and add various negative numbers to it noting that the sum gets increasingly smaller as the addition proceeds from adding negative one through negative four.

Session 3

Free-play.

Teacher: "Make any zero assembly. Take some part of it away. Say what you started with, what you took away, and the final result."

"Can you pretend to have a zero, take something away from it, and know your result?"

"If I say take away negative nine from zero which one will you make and use?"

"If I say take away $^{+}4$ from zero which one will you use now?"

"Can you pretend to have the zero you need to take away negative twelve?" Etc.

"Can you always make the zero you need when you want to take something away?"

Free-play.

"Can you make a train of positive five and zero?"

"How did you make your zero? Does it matter how you make it? What is the value of the train you have? Will it be five with any zero? Take a positive eight rod, is there a way to use what we know about zero to set up the possibility of taking away negative three from positive eight?"

It is helpful to give the students time to experiment with whatever possibilities they find. Once a method is found the students may work in pairs or groups giving one another challenges. This activity gives them practice in adapting a situation to permit the possibility of acting on it. This work should be followed by another session of free exploration. After the experiences outlined above, the study of multiplication can begin. If the students are familiar with the Cuisenaire rods, they can go directly to the study of a positive times a negative. If they have not had much work with the Cuisenaire rods, they should order trains such as five positive threes.

Session 4

Free-play.

Teacher: "Make a train using four negative threes; what shall we call it? Can you find other trains of negative rods which will be the same length as this one?"

The students will probably find all of them; two negative sixes, three negative four's, six negative two's. The last, six negative two's, might require that they work together if they only have four negative two rods. Have the students make a variety of trains of their choice.

"What would you find if you made a train of ten negative fives?" Etc.

Free-play.

Teacher: "Let's make a collection of zeros, all that we can, and set them in front of us. What is the value of this assembly?"

"Find a way to take away two negative fours. What is the value now?"

The students might hesitate at first regarding the removing of the negative fours from atop the positive fours but given time they will come to it. Certain students might take away the zero that is four centimeters long, if they do we ask them, what have you taken away?

"Take away three negative fives, what is the result? Take away three positive threes, what is the result? Can you pretend to take away six negative tens, what would the result be? What else can you pretend to take away, who can make some statements? Is it possible to add a train to this assembly? For example, could you add five negative seven's, what would be the result? What about six positive threes?"

At this point it is useful to return to the acts of the imagination asking them to take away or add any quantities.

They should then begin to challenge one another. Where there arises a difference in opinion, the actual act can be performed to establish the correct result. This session should be followed by free exploration.

Session 5

Free-play.

Teacher: "Make a train of negative twelve. Negative twelve has how many negative two's?"

Most students will make an equivalent train using negative two rods and count them. However, other possibilities might be found; we have to be sure to ask, who can do it another way?

"Negative twelve has how many negative three's? Negative twelve has how many negative sixes?"

"Pretend you have a train of negative thirty. Negative thirty has how many negative tens? Negative thirty has how many negative fives?"

The opportunity that arises here to relate the "has how many" (division) operation with the multiplication should not be overlooked. Questions such as: "If negative thirty has six negative fives, is it also true that six negative fives can be renamed as negative thirty?" We are now ready to work on a number divided by a signed number which sign is different from the dividend. In this operation it will probably be necessary to do more teaching than with any of the others.

Teacher: "Make a train of positive twelve. Positive twelve has how many negative fours?"

Many students will respond, none! The response to that is accepting but it is helpful to ask for open-mindedness as the possibility of using the collection of zeros and performing an action on a portion of it is presented.

Teacher: "Is there anything you could do with certain of those zeros which would produce a train of positive twelve which somehow tells us about negative three?"

If a student sees that he could take away four negative threes to get the equivalent train that gift is accepted and work proceeds from there. However, it is more likely that it will be necessary to direct the perception of the students to that possibility. Once that is accomplished the other divisions follow easily.

"Negative fifteen has how many positive fives? What zeros did you select? What did you have to do to them?"

"Is it true that if we take away three positive fives we will get the result negative fifteen? Do you see the relationship?"

This concludes the outline of experiences on the verbal level for positive and negative integers. As noted earlier, it is not necessary to delay the introduction of writing until all the operations are in hand. Many teachers will want to introduce the writing at the conclusion of the physical and verbal experiences that relate to each new operation. What follows is the set of vocabulary used with the common signs and their contexts. Following the presentation of the vocabulary some of the ways to introduce the writing and at the same time provoke some general statements regarding negative numbers will be presented.

Symbols

Operation	Sign	Decoding
addition	+	"add" or "plus"
subtraction	-	"take away" or "subtract"
multiplication	x	silent, use apostrophe s with second factor
division	÷	"has how many"
	+	(with a rod or numeral) "positive"
	-	(with a negative rod or a numeral) "negative"
	-	(with an unknown) "opposite of"

Session 6

Free-play.

Inasmuch as the students already have the names positive and negative for the respective materials it is possible to begin immediately with an addition problem.

Teacher: "Pick up a negative six rod. Here's how we'll write it, $(^{-}6)$."

"Who can pick up this rod, $(^{-}9)$? Etc."

"What does this say, ($^{-}14$)?"

"Pick up a positive three rod, we will write it like this, ($^{+}3$)."

"What is the opposite of ($^{+}6$)? What is the opposite of ($^{-}9$)? Do you remember what happens when we add opposites? Let's try some. $^{-}9 + {}^{+}9 =$ etc. Can you write some examples? Use large numbers, you don't have to use the rods."

Free-writing. (Note: Many students truly enjoy writing a large variety of equations to zero, especially using large numbers.)

"Let's make some trains using negative rods. How shall we record them?"

Students are usually able to generate the correct writing on their own. Ask them to make a variety of trains with more than just two rods in them and then ask them to imagine the trains that they will write. It is easy to go from here to all of the other addition types. That is, negative plus positive, positive plus negative, positive plus positive, and then additions using more than two addends. The students' attention should be directed toward finding a variety of ways of rewriting an addition with a large set of addends. These possibilities can be experienced by making a train, for example, $^{-}6 + {}^{+}3 + {}^{-}8 + {}^{+}10 + {}^{-}2$. If one adds from left to right, or top to bottom, he does not see an easy way which is to add in this order: $^{-}8 + {}^{-}2 + {}^{+}10 + {}^{-}6$

$+ {}^{+}3 = 0 + {}^{-}6 + {}^{+}3$. Following these experiences a worksheet for class time or homework may be assigned. It can include all of the work to this point and should have space on it for free-writing by the student.

Session 7

Free-play.

Teacher: "Do you remember how to make a zero? Make one and take something away from it, writing everything you do and the final result."

Students may work at a board to write and speak at the same time so that we might see: $0 - {}^{+}3 = {}^{-}3$, and $0 - {}^{-}3 = {}^{+}3$; and hear, "zero take away positive three equals negative three"; and, "zero take away negative three equals positive three."

"Can you imagine any zero. Think of several zeros and perform the same operation. How many can you write?"

Free-writing.

"Now we're ready to use whatever zero we need. From (${}^{+}9$), (written) we want to take away (${}^{+}9$ -) negative three, (${}^{+}9 - {}^{-}3 =$)."

The students will likely be able to do the operation with ease and see immediately that the result is positive twelve. If it is desired, the whole operation can be written thus:

$$
\begin{aligned}
{}^{+}9 - {}^{-}3 &= {}^{+}9 + 0 - {}^{-}3 \\
&= {}^{+}9 + ({}^{+}3 + {}^{-}3) - {}^{-}3 \\
&= {}^{+}9 + {}^{+}3 + ({}^{-}3 - {}^{-}3) \\
&= {}^{+}9 + {}^{+}3 + 0 \\
&= {}^{+}12
\end{aligned}
$$

Likewise when we are subtracting a positive from a negative there is a similar writing.

$$
\begin{aligned}
{}^{-}5 - {}^{+}6 &= {}^{-}5 + 0 - {}^{+}6 \\
&= {}^{-}5 + ({}^{-}6 + {}^{+}6) - {}^{+}6 \\
&= {}^{-}5 + {}^{-}6 + ({}^{+}6 - {}^{+}6) \\
&= {}^{-}5 + {}^{-}6 + 0 \\
&= {}^{-}11
\end{aligned}
$$

It will probably not be helpful to require the student to use this writing in the solving of equations. It should be considered explanatory. The student will be able to image the differences because he has had physical experiences with them. A set of understandings are now available which can be practiced with worksheets and free-writing. If the content of a textbook is a concern the exercises within it may be incorporated into various activities to establish the interrelationship among these activities and those called for by the text.

Session 8

Free-play.

Teacher: "Let's make a train of two negative sixes. This is how we will write it: 2 x $^{-}6$. Did we need to read the 'x'? How shall we read this one, 3 x $^{-}4$?"

"Can you make it?"

"Can you pretend to make the following, 6 x $^{-}5$? What would the result be? What others can you think of?"

"Do you remember the big collection of zeros? We were taking away from them. Let's try one and then we will write what we have done. Take away three negative fives. Do you now have positive fifteen? We will write that in this way. The first sign will tell us 'take away' so we write (-); then how many to take, ($^{-}3$); then the times sign which we don't need to say, ($^{-}3$ x) and finally the sign and the numeral that we will work on ($^{-}3$ x $^{-}5$). Can you read this one in the same way, ($^{-}2$ x $^{-}9$)? What happens when we take away some number of a negative quantity? Will that always be true? So that any multiplication where we have (-) x (-) will equal a positive, (+)."

"What about taking away, (-) a positive, (+)? What will the result of that be? Try some then write them."

"What about adding either positives or negatives, this is just like what we did earlier, making the train, $^{+}6$ x $^{-}3$, or

> ${}^{+}3 \times {}^{-}5$, read, 'add six negative threes, or add three negative fives.'"

This could be the place to establish division as an obvious operation that follows from multiplication. However, the train making which is part of the division operation might help many students come to their own conclusions. Before going to division the students should have the opportunity for free writing. Also, it might be useful to work on multiplication with more than two factors. This is easily accomplished when one realizes that multiplication is a binary operation. That is, if the first two factors are read in the described manner they can be simplified so that that product becomes the multiplier for the next factor. For example:

> ${}^{-}6 \times {}^{-}3 \times {}^{-}4$, we read, "take away six negative threes," that gives positive eighteen. "Add eighteen negative fours," that gives negative seventy-two. Written: $({}^{-}6 \times {}^{-}3) \times {}^{-}4 = {}^{+}18 \times {}^{-}4 = {}^{-}72$.

Session 9

Free-play.

Teacher: "Do you remember this one? ${}^{-}12 \div {}^{-}3 = ?$ We read it, 'Negative twelve has how many negative threes?' What other statements like that one can you make?"

"Do you remember what we had to do when we wanted to know how many positives were in a

> negative, like $^{-}15 \div {}^{+}5$?"
>
> "Do you see that this is similar to the multiplications where we took away three positive fives to make negative fifteen?"

Recommendations

That concludes the presentation of one way to work with students in the study of negative numbers using Cuisenaire rods and the negative material. Obviously these activities will need to be adapted in other situations. The requirements of particular curricula and the learning styles of particular students cannot be predicted. Moreover, while this way of working with negative numbers will likely enhance the student's understanding of them, research into that possibility is an obvious next step. The difficulties outlined by Williams[1] must somehow be obviated in the design of such research.

Other possibilities for next steps include the development of a set of principles for working with signed fractions. Those teachers and supervisors who are familiar with the use of the rods will have little trouble in generating some possible methods for this study. Those who are interested in this possibility should examine Jessica Davidson's, Using the Cuisenaire Rods.[2] Yet another use for the negative materials could be developed by applying them to the "L" assembly that is used for exponential notation by Goutard.[1] That is, where a positive rod represents a positive power, a negative rod could represent a negative power.

---

[1] Williams, "Some Problems Involved in the Experimental Comparison of Teaching Methods," pp. 157-191. See chap. III, p. 76, above.

[2] Jessica Davidson, Using Cuisenaire Rods (New Rochelle, New York: Cuisenaire Company of America, 1969), pp. 36-37; 46-50; 54-57.

Teachers and supervisors who have contact with young children might try using the negative rods as subtraction rods which, by covering lengths, would serve to take away that part that it covered. This use of them might aid in separating the idea of zero from the idea of nothingness.

The best ideas for using the material will arise while they are at work in the classroom.

Conclusion

In this effort to design a concrete manipulative for use in teaching negative numbers relevant theory, practice, and research has been presented. It was possible to define, to some extent, what a manipulative material should be, and it was found that present practice in teaching negative numbers does not use materials that meet the implications of theory. The study continued with an examination of the theory and research pertaining to the Cuisenaire rods and the rods were accepted as a viable product capable of being extended into the set of negative integers. The design was then established as the opposite material to the rods and was therefore defined as negative material. The goal throughout this study has been to create a material which serves to make the mathematics it represents clearer to the students who are trying to learn it. We believe we have achieved this goal.

---

[1] Goutard, Experiences With Numbers in Color, pp. 122-129.

The implications of this achievement are that many more students will be able to grasp the operations involving signed numbers. As noted earlier, all children have not achieved a level of mental operations which permits a totally symbolic introduction to negative numbers. Therefore it cannot be expected that these same children will be able to cope with introductory algebra without some relevant physical experiences with negative numbers. If they have not had the experiences, they will either drop out of the course or they will have difficulty passing it because they will not be able to reflect on the operations. The material which has been presented in this study can help to alleviate the problem because it provides a basis for reflection. And, by alleviating the problems associated with negative numbers, most, if not all, of the content of elementary algebra should become available to students with average ability. In other words, every student who has succeeded in the study of arithmetic will be enabled to continue his mathematical growth through the study of algebra. This is important for the student because it is within the study of algebra that the disparate algorisms of arithmetic are synthesized such that mathematical thought ensues. The act of making known the unknown can be seen as a powerful, creative one and should be available to as many students as possible. This study is one effort in that direction.

Bibliography

Ashlock, Robert B. and Herman, Wayne L. Jr., eds. Current Research in Elementary School Mathematics. New York: Macmillan, 1970.

Ashlock, Robert B. and West, Tommie A. "Physical Representations for Signed-Number Operations." Arithmetic Teacher 14 (November 1967): 549-554.

Association of Teachers of Mathematics. Mathematical Reflections. Cambridge, England: Cambridge University Press, 1970.

________. Notes on Mathematics in Primary Schools. Cambridge, England: Cambridge University Press, 1967.

Aurich, Sister Mary Richard. "A Comparative Study to Determine the Effectiveness of the Cuisenaire Method of Arithmetic Instruction with Children at the First Grade Level." M.A. dissertation, Catholic University of America, 1963.

Biggs, Edith E. and MacLean, James R. Freedom to Learn. Ontario: Addison-Wesley Ltd., 1969.

Biggs, J. B. Mathematics and the Conditions of Learning. Slough, England: National Foundation for Educational Research in England and Wales, 1967.

Brownell, William A. Arithmetical Abstractions; The Movement Toward Conceptual Maturity under Differing Systems of Instruction. Berkeley and Los Angeles: The University of California Press, 1967.

Bruner, Jerome S. The Process of Education. New York: Vintage Books, 1960.

________. Toward a Theory of Instruction. New York: W. W. Norton, 1968.

Cambridge Conference on School Mathematics. "Curriculum for the Elementary School (K-6)." In Problems in the Teaching of Elementary School Mathematics, pp. 8-19. Edited by Klaas Kramer. Boston: Allyn and Bacon, 1970.

________. Goals for School Mathematics. Boston: Houghton Mifflin, 1963.

Canadian Council for Research in Education. *Canadian Experience with The Cuisenaire Method*. Ottawa: The Council, 1964.

Chambers, C. E. *The Cuisenaire-Gattegno Method of Teaching Mathematics*. Reading, England: Educational Explorers, 1964.

Cochran, Beryl S. "Children Use Signed Numbers." *Arithmetic Teacher* 13 (November 1966): 587-588.

Cohen, Louis S. "A Rationale in Working With Signed Number - Revisited." *Arithmetic Teacher* 13 (November 1966): 564-567.

Committee on the Undergraduate Program in Mathematics. *Recommendations on Course Content for the Training of Teachers of Mathematics*. Berkeley, California: Mathematical Association of America, 1971.

Coon, Louis H. "Number Line Multiplication for Negative Numbers." *Arithmetic Teacher* 13 (March 1966): 213-217.

Cotter, Stanley. "Charged Particles: A Model for Teaching Operations with Directed Numbers." *Arithmetic Teacher* 16 (May 1969): 349-353.

D'Augustine, Charles H. *Multiple Methods of Teaching Mathematics in the Elementary School*. New York: Harper and Row, 1968.

Davidson, Jessica. *Using Cuisenaire Rods*. New Rochelle, New York: Cuisenaire Company of America, 1969.

Davis, Robert B. *The Changing Curriculum: Mathematics*. Washington, D. C.: Association for Supervision and Curriculum Development, NEA, 1967.

________. *Explorations in Mathematics: A Text for Teachers*. Palo Alto, California: Addison-Wesley Publishing Co., 1967.

Demchik, Virginia C. "Integer Football." *Arithmetic Teacher* 20 (October 1973): 487-489.

Dienes, Z. P. *Building Up Mathematics*. London: Hutchinson Educational, 1960.

________. "Multi-base Arithmetic." In *Problems in the Teaching of Elementary School Mathematics*, pp. 303-308. Edited by Klaas Kramer. Boston: Allyn and Bacon, 1970.

________. "Some Basic Processes Involved in Mathematics Learning." In Research in Mathematics Education, pp. 21-34. Edited by National Council of Teachers of Mathematics. Washington, D.C.: National Council of Teachers of Mathematics, 1967.

Education Development Center. "A Mathematics Curriculum for K-6." In Current Research in Elementary Mathematics, pp. 286-297. Edited by Robert B. Ashlock and Wayne L. Herman, Jr. New York: Macmillan, 1970.

________. Man: A Course of Study, vol. 5, Baboons. Washington, D.C.: Curriculum Development Associates, Inc., 1970.

Eicholz, Robert E.; O'Daffer, Phares G.; and Fleenor, Charles R. Investigating School Mathematics. Reading, Massachusetts: Addison-Wesley, Inc., 1973.

Ellis, E. N. "An Overview of the Use of the Cuisenaire Arithmetic in British Columbia." In Canadian Experience with The Cuisenaire Method, pp. 3-12. Edited by Canadian Council for Research in Education. Ottawa: The Council, 1964.

________. "The Vancouver Experiments." In Canadian Experience with The Cuisenaire Method, pp. 13-23. Edited by Canadian Council for Research in Education. Ottawa: The Council, 1964.

Emmerling, Frank C., Developer. Individualized Mathematics System. Durham, North Carolina: Center for Individualized Instructional Systems, National Laboratory for Higher Education, 1971.

Fehr, Howard F. and Philips, Jo McKeeby. Teaching Modern Mathematics in the Elementary School. Reading, Massachusetts: Addison-Wesley, 1972.

Feldman, Leonard, ed. Mathematical Learning. New York: Gordon and Breach, 1969.

Fennema, Elizabeth. "Manipulatives in the Classroom." Arithmetic Teacher 20 (May 1973): 350-352.

Frederique. Mathematics and the Child 1. Montreal: Algonquin Publishing, 1971.

Fremont, Herbert. "Pipe Cleaners and Loops -- Discovering How to Add and Subtract Directed Numbers." Arithmetic Teacher 13 (November 1966): 568-572.

Gagnon, Roger; Gelinas, Anselme; Gignac, Lucien; and Trepanier, François. "Etude Comparative -- Méthode Cuisenaire -- Méthode Traditionelle." Translated by John Trivett. Montreal, 1964. (Mimeographed.)

Gattegno, Caleb. For The Teaching of Mathematics. 3 vols. Reading, England: Educational Explorers, 1963.

________. Mathematics With Numbers in Color. 4 vols. Mount Vernon, New York: Cuisenaire Company of America, 1963.

________. Modern Mathematics With Numbers in Colour. Reading, England: Educational Explorers, 1963.

________. Now Johnny Can Do Arithmetic. Reading, England: Educational Explorers, 1963.

________. What We Owe Children. New York: Outerbridge and Dienstfrey, 1970.

Glennon, Vincent J. and Callahan, Leroy G. Elementary School Mathematics A Guide to Current Research. Washington, D.C.: Association for Supervision and Curriculum Development, NEA, 1968.

Goutard, Madeleine. Experiences With Numbers in Color. New Rochelle, New York: Cuisenaire Company of America, 1974.

________. Mathematics and Children. Reading, England: Educational Explorers, 1964.

________. Talks for Primary School Teachers. Mount Vernon, New York: Cuisenaire Company of America, 1964.

Grossnichle, Foster E. and Bruechner, Leo J. Discovering Meanings in Elementary Mathematics. New York: Holt, Rinehart and Winston, 1963.

Hall, Wayne H. "These Three: Minus, Negative, Opposite." Arithmetic Teacher 21 (December 1974): 712-713.

Haynes, Jerry Oscar. "Cuisenaire Rods and the Teaching of Multiplication to Third-grade Children." Ph.D. dissertation, Florida State University, 1963.

Henderson, Kenneth B. Teaching Secondary School Mathematics. Washington, D.C.: Association of Classroom Teachers of the NEA, 1969.

Hill, Warren H., Jr. "A Physical Model for Teaching Multiplication of Integers." Arithmetic Teacher 15 (October 1968): 525-528.

Hollis, Loye Y. "A Study to Compare the Effects of Teaching First and Second Grade Mathematics by the Cuisenaire-Gattegno Method with a Traditional Method." School Science and Mathematics 65 (November 1965): 683-687.

Intermediate Mathematics Methodology Committee. Intermediate Mathematics Methodology. Toronto, Ontario: Ontario Institute for Studies in Education, 1968.

Jacobs, Harold R. Mathematics: A Human Endeavor. San Francisco: W. H. Freeman, 1970.

Jencks, Stanley M. and Peck, Donald M. Building Mental Imagery in Mathematics. New York: Holt, Rinehart and Winston, 1968.

Karatzinas, D. and Renshaw, T. "Testing the Effectiveness of the Cuisenaire Material." Scottish Educational Journal (October 1958): 613.

Keislar, Evan R. "Abilities of First Grade Pupils to Learn Mathematics in Terms of Algebraic Structure by Teaching Machines." Los Angeles: The Cooperative Research Program U.S.O.E. Project SAE 8898 No. 1090, University of California, Los Angeles, 1961.

Kidd, Kenneth P.; Myers, Shirley S.; and Cilley, David M. The Laboratory Approach to Mathematics. Chicago: Science Research Associates, 1970.

Kingston, J. Maurice. Mathematics for Teachers of the Middle Grades. New York: John Wiley and Sons, 1966.

Kramer, Klaas, ed. Problems in the Teaching of Elementary School Mathematics. Boston: Allyn and Bacon, 1970.

Kramer, Klaas. Teaching Elementary School Mathematics. Boston: Allyn and Bacon, 1970.

Kunz, John. Director of Education, Cuisenaire Company of America, New Rochelle, New York. Interview, 17 October 1974.

Land, Frank W., ed. New Approaches to Mathematics Teaching. London: Macmillan and Company, 1963.

Lucow, William H. "Cuisenaire Method Compared With Current Methods of Teaching Multiplication and Division in Grade 3." Winnipeg, Manitoba: The Manitoba Teachers' Society, 1962.

Magnuson, Russell C. "Signed Numbers." Arithmetic Teacher 13 (November 1966): 573-588.

McLellan, James A. and Dewey, John. The Psychology of Number. New York: D. Appleton and Company, 1914.

National Council of Teachers of Mathematics. Thirtieth Yearbook: More Topics in Mathematics. Washington, D.C.: National Council of Teachers of Mathematics, 1969.

Page, David A. Number Lines, Functions, and Fundamental Topics. New York: Macmillan, 1964.

Passy, Robert Albert. "How Do Cuisenaire Materials in a Modified Elementary Mathematics Program Affect the Mathematical Reasoning and Computational Skill of Third-grade Children?" Ph.D. dissertation, New York University, 1963.

Peterson, John C. "Fourteen Different Strategies for Multiplication of Integers or Why $(^-1)(^-1)=^+1$." Arithmetic Teacher 19 (May 1972): 396-403.

Piaget, Jean. Genetic Epistemology. Translated by Eleanor Duckworth. New York: Columbia University Press, 1970.

________. Psychology and Epistemology. Translated by Arnold Rosin. New York: Viking Press, 1971.

________. Six Psychological Studies. Translated by Anita Tenzer. Edited by David Elkind. New York: Vintage Books, 1967.

Pratt, Edna M. "A Teaching Aid for Signed Numbers." Arithmetic Teacher 13 (November 1966): 589-590.

Root, Lawrence T. "One School's Experience with Cuisenaire-Gattegno Mathematics Instruction." New York State Mathematics Journal 13 (January 1963): 24-27.

Scandura, Joseph M., ed. Research in Mathematics Education. Washington, D.C.: The National Council of Teachers of Mathematics, 1967.

School Mathematics Study Group. Studies in Mathematics, Volume 6, Number Systems. New Haven, Connecticut: Yale University, 1961.

Schultz, James E. "Why I Don't Have Any Examples of Negative Numbers." Arithmetic Teacher 20 (May 1973): 365.

Science Research Associates. Elementary Mathematics Program, Book 6, Teacher Guide. Chicago: Science Research Associates, 1968.

Servais, W. "The Significance of Concrete Materials in the Teaching of Mathematics." In Mathematical Reflections, pp. 203-208. Edited by Association of Teachers of Mathematics. Cambridge, England: Cambridge University Press, 1970.

Servais, W. and Varga, T. *Teaching School Mathematics*. Middlesex, England: Penguin Books, 1971.

Shah, Sair Ali. "An Experiment With the Cuisenaire Material for Teaching Arithmetic to Observe Organisational Problems, Reactions of Teachers and of Children." St. Augustine, Trinidad: Institute of Education, University of the West Indies, 1968.

Sherman, Helene. *Common Elements in New Mathematics Programs*. New York: Teachers College Press, Columbia University, 1972.

Skemp, Richard R. *The Psychology of Learning Mathematics*. Middlesex, England: Penguin Books, 1971.

Stern, Catherine. *Children Discover Arithmetic: An Introduction to Structural Arithmetic*. New York: Harper and Row, 1949.

Sweeney, John R. "An Experimental Study Comparing the Cuisenaire Method with Traditional Methods in Grade 1 Mathematics." In *Canadian Experience With The Cuisenaire Method*, pp. 117-140. Edited by Canadian Council for Research in Education. Ottawa: The Council, 1964.

Trivett, John V. "The Cuisenaire Rods -- Numbers in Color." *Mathematics Teaching* 18 (1962): 37-41.

University of Illinois Arithmetic Project, David A. Page, ed. *Institute for Elementary School Teachers*. Watertown, Massachusetts: Education Development Center, 1968.

Wheeler, Ruric E. *Modern Mathematics: An Elementary Approach*. Belmont, California: Brooks/Cole Publishing, 1970.

Willerding, Margaret F. *Elementary Mathematics, Its Structure and Concepts*. New York: John Wiley and Sons, 1966.

Williams, John D. "Some Problems Involved in the Experimental Comparison of Teaching Methods." In *Mathematical Learning*, pp. 157-191. Edited by Leonard Feldman. New York: Gordon and Breach, 1969.

_______. *Teaching Techniques in Primary Maths*. London: National Foundation for Educational Research in England and Wales, 1971.

________. "The Evaluation of Three Maths Courses." Journal of Structural Learning 3 (1972): 41-79.

Worthen, Blaine R. "A Comparison of Discovery and Expository Sequencing in Elementary Mathematics Instruction." In Research in Mathematics Education, pp. 44-59. Edited by National Council of Teachers of Mathematics. Washington, D.C.: National Council of Teachers of Mathematics, 1967.